THE PRACTICAL APPLICATION OF VARIANCE REDUCTION TECHNIQUES IN PROBABILISTIC ASSESSMENTS

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ABSTRACT

The use of variance reduction techniques in probabilistic risk assessments is examined, using an actual assessment model as an example. Variance reduction techniques may allow probabilistic risk assessments to be performed more efficiently. The efficiencies of stratified sampling, Latin hypercube sampling, and antithetic sampling are compared with simple random sampling. Stratified sampling appears to be less efficient than simple random sampling, and is not recommended for use in probabilistic assessments. The efficiency improvement offered by antithetic and, particularly, Latin hypercube sampling can justify their use under certain circumstances.

INTRODUCTION

Probabilistic risk assessments are widely used in the nuclear industry, both in the areas of reactor safety (1)(2), and in waste management. (3)(4) The use of variance reduction techniques in probabilistic risk assessments is examined, using an environmental assessment model as an example. Such techniques can allow probabilistic risk assessments to be performed more efficiently. The example used in this paper is the radon-dose model used in an assessment of uranium mine tailings. (5)

The rationale for performing a probabilistic assessment (and hence, basing the performance objective on risk) is to avoid the absolutism of a 'worst-case' analysis. In the context of waste management a treatment or disposal concept could be rejected outright by a high dose from a 'worst-case' analysis, even if the probability of that worst possible consequence is very low. Such problems do not arise if the performance assessment is based on combined probability-consequence (risk) information. (6) However, to perform a probabilistic risk assessment, it is necessary to know the risk associated with radiological exposures.

In the range of very high annual dose equivalents (above about 1 $Sv.a^{-1}$) a serious health effect (acute illness or death) will almost certainly result to an individual irrespective of the actual dose level. Such high dose rates will not, however, occur in the disposal or treatment of uranium mine tailings. Hence, acute health effects are not considered further in this paper.

In the range of doses likely to be encountered in the treatment or disposal of uranium mine tailings an exposed individual may, or may not, experience a deleterious health effect. The probability of a stochastic health effect is proportional to the dose level. The constant of proportionality (γ) that relates the health effect probability to the dose level is approximately 2×10^{-2} Sv^{-1}. (7) Using this value for the proportionality constant ensures the inclusion of both somatic and genetic effects. Thus, the radiological risk to an individual is given by

$$R = \gamma\int_0^\infty p(H)\ H\ .dH \qquad (1)$$

where, R is the radiological risk to an individual (health effects per annum), H is the dose level (annual dose equivalent in $Sv.a^{-1}$), and p(H) is the probability density function (PDF) of annual dose equivalents.

Now, from the mathematical properties of probability density functions,

$$R = \gamma\int_0^\infty p(H)\ H\ .dH = \gamma\bar{H} \qquad (2)$$

where $\bar{H}$ is the mean annual dose. Thus the calculation of the annual risk reduces to the calculation of the product of γ and the mean annual dose.

A condition for either a treatment or disposal option to be judged acceptable is that the calculated risk should be below the annual risk objective. In general, the risk will vary as a function of time. The variation of the risk with time is not discussed in this paper, but it is implicit that the annual risk objective must be satisfied at all times.

An annual individual risk objective for a single management site of the order of 10^{-6}, and associated probabilistic methodology, is approved in many countries and is currently the subject of regulatory consultation in Canada. (8)

PROBABILISTIC METHODOLOGY

In a probabilistic risk assessment, parameters are assigned probability density functions on the basis of current technical knowledge. Parameter values are sampled from their respective PDFs, and the annual dose equivalent for each realization (H_i) is calculated. The estimate of the mean annual dose ($E(\bar{H})$) is the mean annual dose obtained from many realizations. It is not possible *a priori* to calculate how far the estimate $E(\bar{H})$ will be from the true mean $\bar{H}$. The convergence characteristics of $E(\bar{H})$ with respect to the number of realizations must be investigated.

In the case of simple random sampling the central limit theorem allows confidence bounds to be placed on $E(\bar{H})$ provided that sufficient realizations have been performed. In general, the standard deviation of $E(\bar{H})$ is equal to $\sigma_H/\sqrt{n}$, where σ_H is the standard deviation of the values of H_i obtained from the n realizations. The convergence of the risk estimate, for the radon dose model, is shown in Figure 1. The 95% confidence limits shown in Figure 1 are derived using the central limit theorem.

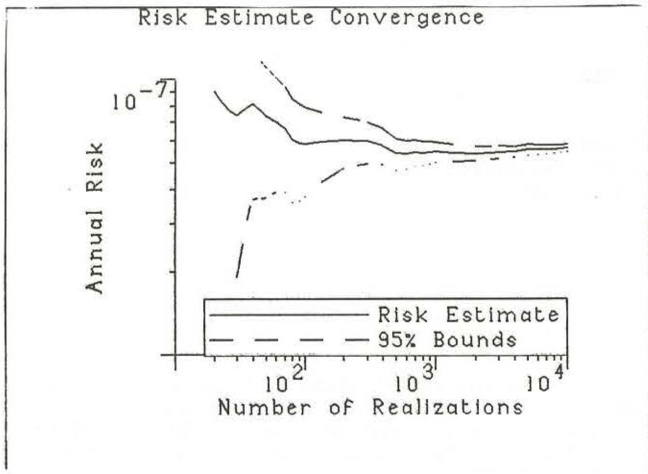

FIGURE 1: RISK ESTIMATE CONVERGENCE

The value of σ_H is determined by the model and the parameter PDFs. Thus, the estimate of mean dose rate can only be improved by performing greater numbers of realizations. Since the estimate accuracy is proportional to $\sqrt{n}$, large amounts of computer time may be necessary if very accurate estimates are required.

The aim of variance reduction techniques is to reduce the variance in the H_i, and obtain a more accurate value of $E(\bar{H})$ with a smaller number of realizations.

VARIANCE REDUCTION TECHNIQUES

In general, variance reduction is a technique where the known information about a system is used to gain further information about the system. (9) At the extreme of knowing nothing about a system, then no variance reduction can be achieved. At the other extreme of complete knowledge of a system, the variance is already zero and there is no need for further work.

Variance reduction may be achieved in individual systems by making use of knowledge of the physico-chemical processes that comprise the system. No general variance reduction technique can, however, be applied to different physico-chemical systems, since a specific knowledge of each individual system is required. However, the PDFs are always known, and it is the sampling from the PDFs that provide a general opportunity for variance reduction.

Several sampling techniques have been proposed to reduce the variance. The most commonly proposed variance reduction techniques are stratified sampling (10), Latin hypercube sampling (11), and the use of antithetic variables. (12)

Stratified Sampling

In stratified sampling each of the parameter PDFs is partitioned into a number of intervals. For an unbiased result to be obtained, each of the intervals must be of equal probability. Parameter values are selected from these intervals with the aim of ensuring that the full range of possible parameter values is sampled in an efficient manner. Every combination of PDF intervals is used in calculating $E(\bar{H})$. Stratified sampling is, therefore, related to a full factorial experimental design.

As an example, consider the case where the dose equivalent is a function of two variables (x,y),

$$H = f(x,y) \quad , \qquad (3)$$

and the PDFs representing x and y are each partitioned into three equal-probability intervals. Then the estimate of $E(\bar{H})$ from stratified sampling is given by,

$$E(\bar{H}) = (1/3^2) \sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) \quad , \qquad (4)$$

where x_i is a randomly selected value of x from the ith interval of the PDF of x, and y_j is a randomly selected value of y from the jth interval of the PDF of y. This example is illustrated in Figure 2.

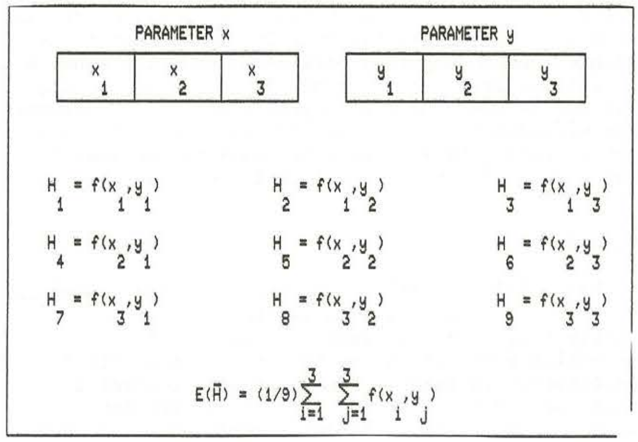

FIGURE 2: EXAMPLE OF STRATIFIED SAMPLING

In general, for m parameters (xi:i=1,2,...,m), each divided into k intervals,

$$E(\bar{H}) = \frac{1}{k^m} \sum_{im=1}^{k} \ldots \sum_{i2=1}^{k} \sum_{i1=1}^{k} f(x1_{i1}, x2_{i2}, \ldots xm_{im}) \qquad (5)$$

It can be seen from Equation 5 that a stratified sampling scheme requires k^m realizations to provide an unbiased value of $E(\bar{H})$. Thus stratified sampling may require a very large number of realizations unless k and m are small. The expense inherent in stratified sampling is the reason for the use of the less expensive Latin hypercube sampling scheme.

Latin hypercube sampling

In Latin hypercube sampling each of the parameter PDFs is partitioned into a number of equal-probability intervals, as is the case with stratified sampling. In stratified sampling every combination of PDF intervals is used in calculating $E(\bar{H})$. In Latin hypercube sampling, by contrast, each PDF interval is used only once in the calculation of E(H).

Consider again the case of H = f(x,y), with the PDFs representing x and y being divided into three equal-probability intervals. The first value of x is made by randomly selecting an interval of x (say interval 2) and randomly selecting a value for x from within that interval. The corresponding value of y is similarly made by randomly selecting an interval of y (say interval 3) and randomly selecting a value for y from within that interval. Thus, in this example, the first realization of H (H_1) is given by,

$$H_1 = f(x_2, x_3) \quad (6)$$

This process is repeated, ensuring no particular interval is used more than once, to produce additional realizations H_i. This example is illustrated in Figure 3.

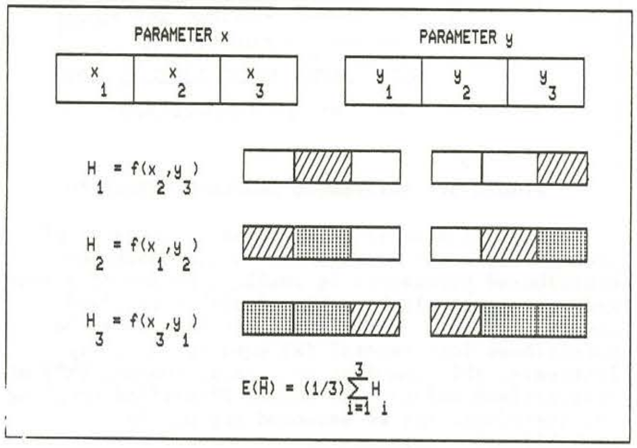

FIGURE 3: EXAMPLE OF LATIN HYPERCUBE SAMPLING

In general, k realizations are required to produce an unbiased value of $E(\bar{H})$, where k is the number of intervals into which each PDF is partitioned. Latin hypercube sampling corresponds, therefore, to a highly fractionated experimental design. Should additional realizations be required the entire Latin hypercube procedure can be repeated using different random selections.

In addition to sampling schemes involving the partitioning of PDFs into equal-probability intervals, variance reduction may be achieved by inducing correlations between selected values. One such scheme of correlation induction is by the use of antithetic variables.

Antithetic sampling

The aim of antithetic sampling is to reduce the variance by minimizing the correlation (maximizing the negative correlation) between pairs of realizations.

If F is the cumulative distribution function of a parameter, with inverse F^{-1} defined as

$$F^{-1}(r) = \inf\{x : F(x) \geq r\} \text{ for } 0 \leq r \leq 1 \quad , \quad (7)$$

then the correlation between two values of x (x, x') is minimized if

$$x = F^{-1}(r) \text{ and} \quad (8a)$$

$$x' = F^{-1}(1-r) \; , \; 0 \leq r \leq 1 \; . \quad (8b)$$

Antithetic sampling, therefore, produces pairs of realizations. The first realization is produced by randomly selecting parameter values from their PDFs (to give parameter value set x_i). The second of the realizations is produced using the complementary values of the first parameter set (x_i').

As an example consider again $H = f(x,y)$ where x is uniformly distributed over the interval (0,1) and y is normally distributed with mean (μ) = 1 and standard deviation (σ) = 2. If $H_1 = f(0.7,3)$ then $H_1' = f(0.3,-1)$.

In practice, many realization pairs are produced, using different random selections, so that the accuracy of $E(\bar{H})$ is improved.

RESULTS

The use of any of the above variance reduction techniques leads to an increase in computer time requirement, relative to simple random sampling, because of the additional record keeping required. However, in most probabilistic assessments, parameter value selection is a minor component of total computer time use. Thus the computer time increase required by these variance reduction techniques is not a significant disadvantage.

A disadvantage of more potential importance is that, unlike simple random sampling, confidence limits cannot be placed ***a priori*** on the accuracy of $E(\bar{H})$. It is possible, however, to mitigate against this disadvantage if resources allow multiple sets of realizations to be performed.

Throughout these simulations a statistically sound pseudo-random number generator (13) was employed to ensure unbiased results. The results using the various variance reduction techniques are compared with a reference value obtained from 10^8 realizations using simple random sampling. The reference value is accurate to a few parts in 10^4 (calculated using the central limit theorem).

Stratified Sampling

The radon dose model (5) has twelve distributed parameters (m = 12). It is not practical to partition each of the twelve parameters into more than two equal-probability intervals (k = 2). Even with only two intervals 2^{12} = 4096 realizations are required to give an unbiased estimate of $E(\bar{H})$.

Since each parameter is partitioned into only two intervals it cannot be expected that the variance is much reduced relative to simple random sampling. This is illustrated in Figure 4 where the relative errors of stratified sampling and simple random sampling are given as a function of number of realizations. As can be seen from the figure, there is little or no improvement relative to simple random sampling.

These results indicate that stratified sampling is impractical to use, since in order to achieve significant variance reduction we must have k>2. Therefore, impractically large numbers of realizations are required.

Latin hypercube sampling

For the Latin hypercube simulations each of the twelve parameters was partitioned into ten equal-probability intervals. Thus multiples of ten realizations are required to produce unbiased estimates.

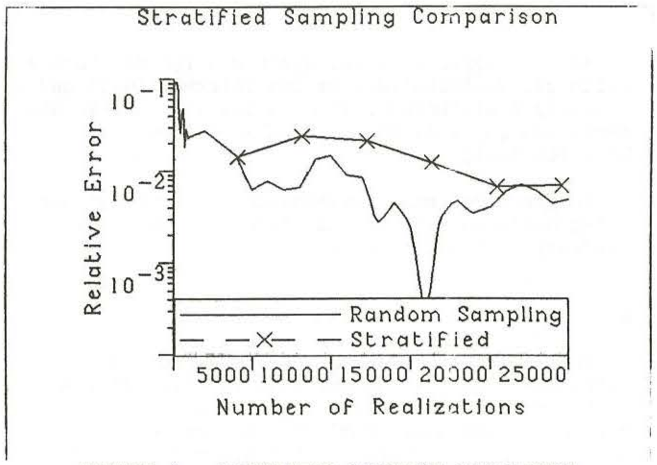

FIGURE 4: STRATIFIED SAMPLING COMPARISON

As can be seen from Figure 5 Latin hypercube sampling produces improved estimates, relative to simple random sampling, at low numbers of realizations. As the number of realizations increases the variance obtained with Latin hypercube sampling approaches that of simple random sampling.

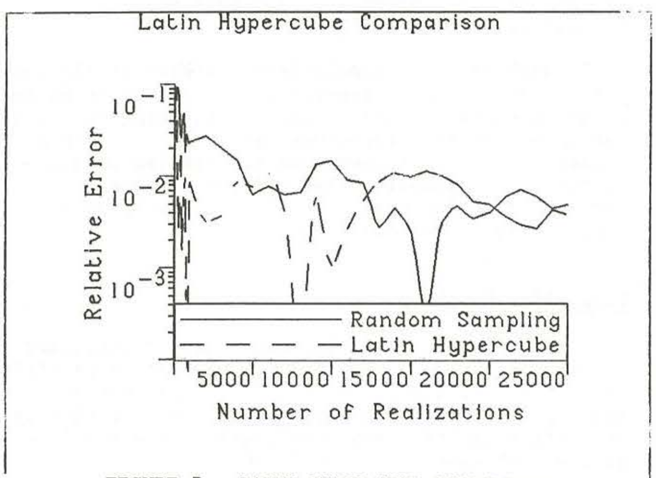

FIGURE 5: LATIN HYPERCUBE COMPARISON

Antithetic sampling

The results for antithetic sampling are shown in Figure 6. The results are similar to those obtained from Latin hypercube sampling, in that there is an accuracy improvement at low numbers of realizations. As the number of realizations increases the variance obtained with antithetic sampling approaches that of simple random sampling.

CONCLUSIONS

Variance reduction techniques can improve the efficiency of probabilistic risk assessments, since they potentially enable improved estimates of mean annual dose (and hence risk) to be made with fewer realizations.

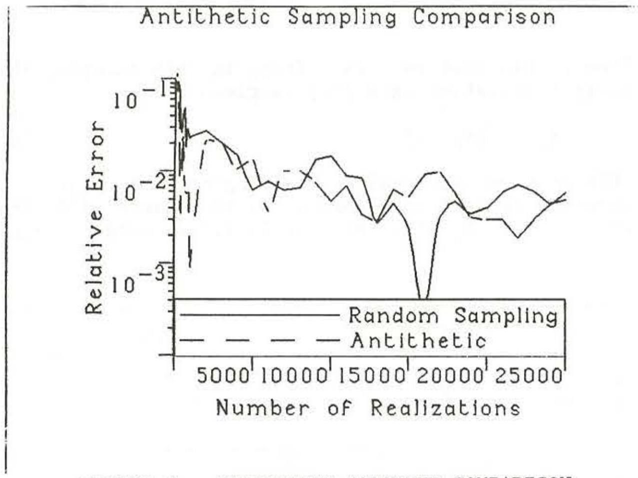

FIGURE 6: ANTITHETIC SAMPLING COMPARISON

The use of stratified sampling is impractical for probabilistic assessments unless the number of distributed parameters is small. Significant variance reduction can only be achieved with stratified sampling if the m distributed parameters can be partitioned into several (k) equal-probability intervals. This requires very large numbers (k^m) of realizations unless m is small. Stratified sampling is, therefore, not recommended for use in probabilistic assessments.

Both Latin hypercube and antithetic sampling show significant variance reduction at low numbers of realizations. Latin hypercube shows marginally more variance reduction. The variance using either Latin hypercube or antithetic sampling converges to that of simple random sampling as the number of realizations increases.

It is not possible ***a priori*** to produce confidence limits on the estimate with either Latin hypercube or antithetic sampling. If very accurate values of the estimate are required, possibly for regulatory purposes, then simple random sampling is recommended since it is possible to produce confidence limits to accompany the estimate.

Latin hypercube, and to a slightly lesser extent antithetic sampling, offer the ability to produce reasonable estimates of mean annual dose (and hence risk) at low numbers of realizations.

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