An Analytical Expression For The True-Value Approach In Extreme-Value Statistics Applied To Channel Powers

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Abstract

A method for evaluating the true value of the maximum channel power, based on a set of measured channel powers, has been proposed in a Canadian Nuclear Society (CNS) conference paper by Paul Sermer *et al.* [1]. This method makes use of the Monte Carlo technique to calculate the extreme value distribution of channel powers and obtain the compliance probability of the true maximum.

This paper describes an analytical expression for the same extreme value distribution, which leads to the same channel power compliance probabilities as are obtained with the Monte Carlo technique. Though the analytical expression must be evaluated numerically to obtain the probability, there is a potential for a significant gain in computing time when compared to the Monte Carlo technique, especially when the method is embedded in complex calculations such as the determination of trip probabilities in CANDU Regional Overpower Protection systems.

1. Introduction

The concept of extreme value statistics (EVS) is based on the notion that the extreme of a population follows a different statistical distribution than the individual members of the population. A simple example is the throwing of a pair of dice: each die by itself throws eyes according to a flat probability distribution between 1 and 6; the (discrete) distribution is symmetric around the average of 3.5. However, if we throw the pair of dice and tally the number of eyes on the die with the maximum eyes in each throw, we can easily see that the distribution is skewed towards the higher eyes; it is not symmetric anymore, and the average is higher than that of the individual dice, namely around 4.5.

This concept has been applied by Sermer et al. [1] to the calculation of the probability that a given observed channel power distribution is in compliance with a power limit *L*.

In general, a single observable like a channel power has a true value p, and an observed value P, related to p by

$$P = p + \varepsilon \,. \qquad (1)$$

Here, ε is the deviation from the true value, which comes from a probability function such as a Gaussian, such that, if many measurements were performed, P would have a Gaussian distribution around p with σ as standard deviation[1].

[1] In some cases, the relation changes to $p = P + \varepsilon$: when a single average value P is obtained from modeling a quantity, and the true values p are expected to be spread around it. Pressure tube creep is an example of such an observable; an average creep has been modeled (as a function of time), and we expect the true values for each channel to be distributed around that value. This case is not further considered in this paper.

Generally, a number of channels, say N, will have similar powers, all close to the maximum. These are called the *contributors* or *participants*. We introduce the vector $\boldsymbol{P} = (P_1, P_2, \ldots, P_N)$ to describe the observed channel powers of the contributors, and $\boldsymbol{p} = (p_1, p_2, \ldots, p_N)$ to describe their true channel powers.

True compliance means that $\max(\boldsymbol{p}) < L$. However, since we only have the observed values $\boldsymbol{P}$, we can assert compliance only with a certain probability. As the number of contributors increases, the chance that one of them has a high reading increases. This is reflected in the EVS, which, as illustrated with the dice, shows that the maximum observed channel power follows from a distribution, which is: a) not Gaussian anymore, b) not symmetric anymore, and c) *has an average that is higher than the true maximum channel power*.

A proper statistical treatment of EVS, based on the True Value Approach (TVA) gives a better estimate of the true maximum channel power than simply adding an error allowance to the observed maximum channel power. Such a treatment has been proposed by Sermer et al. and is being pursued for implementation in ROP set-point determination. The treatment is based on a Monte Carlo (MC) method, described briefly in the following section. The section thereafter gives an analytical expression that yields the same probability. The last section shows a numerical comparison of both methods.

2. Monte Carlo approach to EVS

This section describes the procedure and the equations used in the Monte Carlo (MC) version of the true value approach (TVA) to EVS; it does not provide a theoretical justification of the method. That is described elsewhere in the literature; see Reference 1 and references therein.

The procedure is briefly as follows: the compliance probability is defined as the probability that for a given observed power distribution $\boldsymbol{P}$ of N contributing channels, the true channel power distribution $\boldsymbol{p}$ is in compliance with the limit L, *i.e.* $\max(\boldsymbol{p}) < L$. The non-compliance probability is called α; hence $(1-\alpha)$ is the compliance probability.

An error η is defined as the relative error in the maximum value of $\boldsymbol{P}$:

$$\eta = \frac{\max\{\boldsymbol{P}\} - \max\{\boldsymbol{p}\}}{\max\{\boldsymbol{p}\}}. \tag{2}$$

From η, a compliance uncertainty η_α is derived in such a way that:

$$\text{Prob}\{\eta < -\eta_\alpha\} = \alpha\,, \tag{3}$$

for a given α. A problem with the expression for η is that it is based on knowledge of the true maximum. We do not know this true maximum, so we have to rely on our best estimate of the true maximum, which we call $P_{max} \equiv \max\{\boldsymbol{P}\}$. This is claimed to be a good 'surrogate' as long as the error distributions are realistic. Now, using the Monte Carlo procedure, many sets of N channel powers are generated by 'sprinkling' deviations $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, which are each chosen from the uncertainty distributions for the channel powers, on $\boldsymbol{P}$ to obtain many sets of $\hat{\boldsymbol{P}} = \boldsymbol{P} + \boldsymbol{\varepsilon}$. A new error $\hat{\eta}$ is introduced by:

$$\hat{\eta} = \frac{\max\{\widehat{\boldsymbol{P}}\} - P_{\max}}{P_{\max}}. \tag{4}$$

Thus, for each set of generated $\widehat{\boldsymbol{P}}$, there is one value of $\hat{\eta}$. The distribution of these $\hat{\eta}$ is the EVS error distribution of the surrogate maximum of the channel powers. It is shown in Figure 1, which also shows a possible value of η_α, in this case for $\alpha = 5\%$.

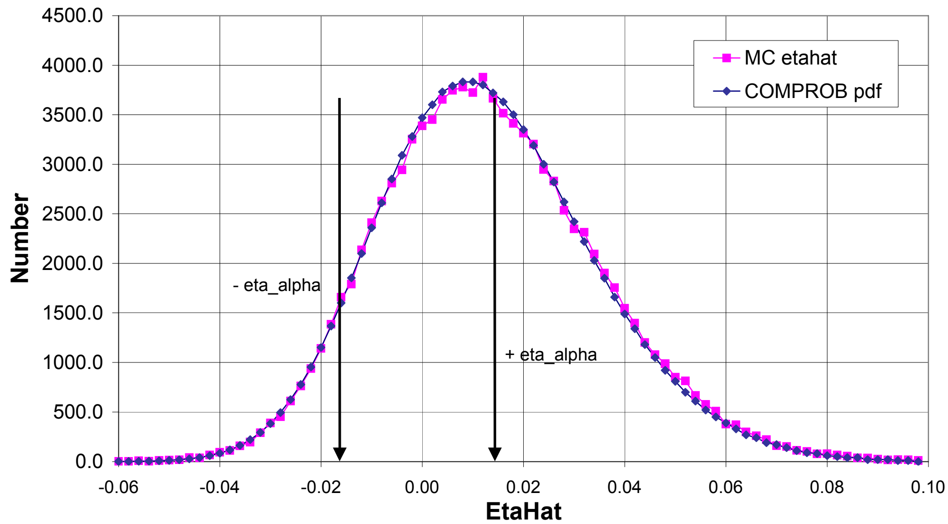

Figure 1. The $\hat{\eta}$ distribution for a set of five contributing channel powers, each with standard deviations of 3%. The pdf derived with COMPROB for the same set is shown for comparison; it is described in Section 3.

A new value η_α is obtained by the requirement that

$$\text{Prob}\{\hat{\eta} < -\eta_\alpha\} = \alpha. \tag{5}$$

The test for compliance with probability (1-α) is now that (without derivation):

$$\text{Accept } \max\{\boldsymbol{p}\} \le L \quad \text{if and only if } \max\{\boldsymbol{P}\} \le L(1-\eta_\alpha). \tag{6}$$

In summary: a given observed $\boldsymbol{P}$ yields a value for η_α at a given probability (1-α) through the EVS distribution for $\hat{\eta}$; the compliance test (6) then determines whether the channel powers are in compliance. Conversely, for given $\boldsymbol{P}$ and L, we can obtain η_α from the equal sign in (6), and derive the probability (1-α). The latter is done in the comparison shown below.

3. The analytical approach to EVS

The traditional approach to the analytical expression for EVS starts with the probability density functions (*pdf*) of the contributors (*e.g.* the Gaussian error functions with mean 0 and standard deviation 1). The *pdf* are numerically integrated to obtain the probability function P_{Gi}:

$$P_{G_i}(x) = \int_{-\infty}^{x} pdf_i(m)dm, \tag{7}$$

The probability of compliance (ComProb) is then given by the following product[2]:

$$\text{ComProb} = \prod_{i=1}^{N}\left(1 - P_{Gi}\left(\frac{(L/P_i)-1}{\sigma_i}\right)\right). \tag{8}$$

The shape of the probability density function derived from ComProb is shown in Figure 1; it is seen to be identical to the MC distribution of $\hat{\eta}$ within the statistical errors inherent to the MC method. However, the value of ComProb obtained from (8) does not take into account that the *observed* maximum channel power follows an EVS distribution derived from *true values*. Rather, the method generates a new EVS distribution derived from observed channel powers and calls it the distribution of the true maximum. Although this is the correct approach for cases described in footnote 1, it is overly conservative in the case of channel power compliance.

In order to account for the fact that the true maximum is most likely lower than the average value of the EVS distribution that the observed maximum follows, we need to modify equation (8) as follows:

$$\text{ComProb TVA} = 1 - \prod_{i=1}^{N}\left(1 - P_{Gi}\left(\frac{(P_{\max}/P_i)-(L/P_{\max})}{\sigma_i}\right)\right). \tag{9}$$

Expression (9) is obtained by changing the argument of P_{Gi} such that the probability density function coincides with the $\hat{\eta}$ distribution described above. Note that if $N=1$, i.e. for a single contributing channel, it reduces to equation (8), as it should. (Considering that $P_G(-m) = 1 - P_G(m)$).

4. Numerical comparison of both methods

A numerical comparison of the methods with existing channel power data is hampered by the fact that we do not know the true values of the channel powers. However, we can easily generate many data sets $\boldsymbol{p}$ which we label as 'true', and then derive 'observed' data sets $\boldsymbol{P}$ by 'sprinkling' errors around it: $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{\varepsilon}$. A large number of such 'true' and corresponding 'observed' data sets have been generated for the comparison. This generation and the analyses compared here were done with a simple FORTRAN 95 program.

We have performed numerical comparisons with several different sets of parameters, but here we show only the comparison of the two methods under the following conditions: the number of participants N is 5, and all σ_i are identical and equal to 3%. The results are shown in Figures 2 and 3, which show the same data; Figure 3 is a close-up of the range above 90%.

[2] This expression is similar to the one implemented in the analytical code currently used to calculate the trip probability of ROP systems.

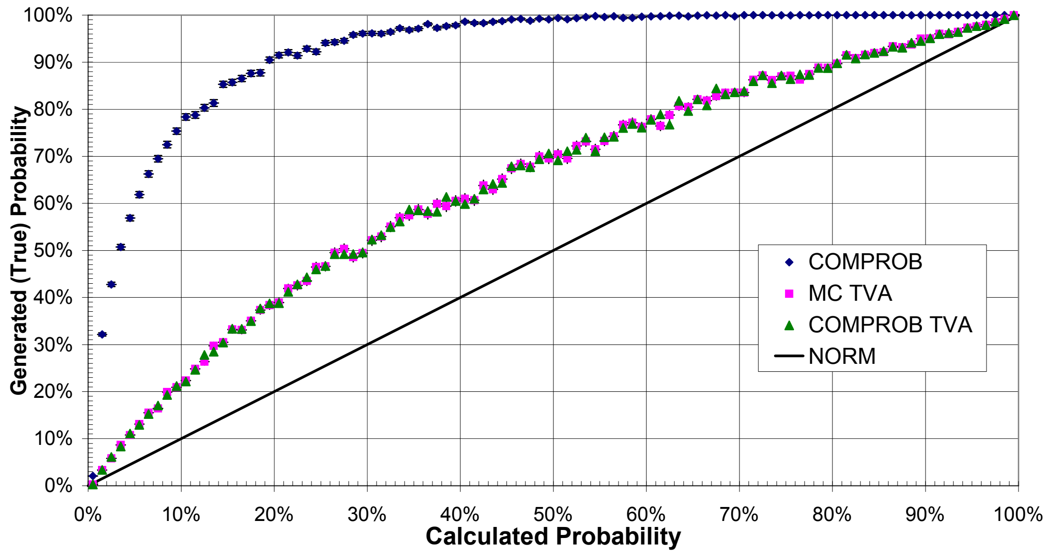

Figure 2: True probability of channel power compliance versus calculated probability of simulated distribution with five contributors and 3% channel power uncertainty.

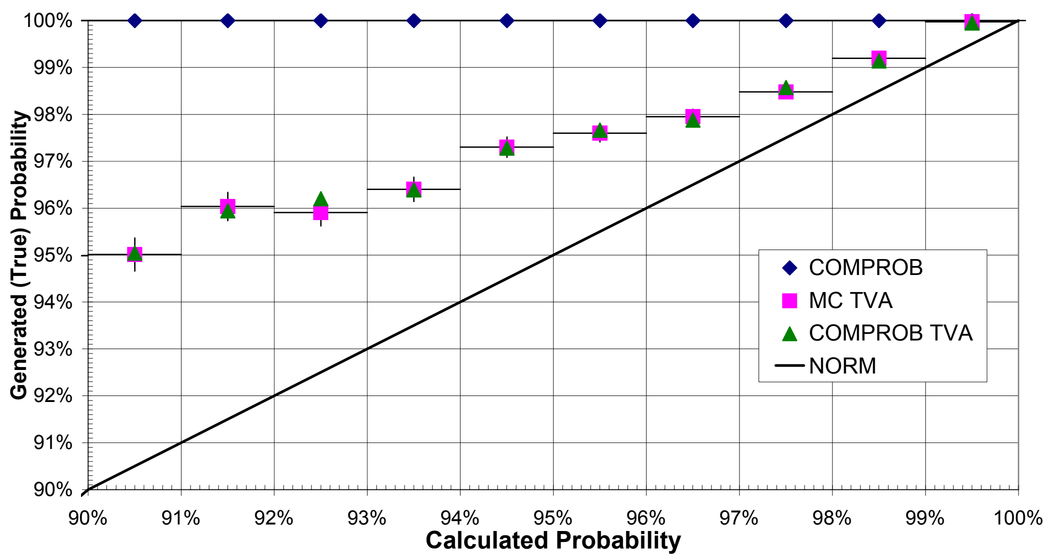

Figure 3: Same as Figure 2, with a close-up of the range between 90% and 100%.

The plots are to be interpreted as follows: each data point, say the square point labeled MC TVA between 92 and 93%, contains all generated configurations for which the MC method predicted a compliance probability (1-α) of between 92 and 93%. Since we know of each configuration whether or not it is compliant (after all we generated it), we can count the percentage of configurations that are truly compliant (~96%). That number is plotted against the *y*-axis as the *true* probability. The same procedure is applied to the analytical method (the triangles labeled COMPROB TVA).

From the figures, we draw two conclusions:

1. The MC-based method and the analytical method give identical results. Slight differences are due to the statistical errors inherent to the MC technique.
2. Both curves lie *above* the diagonal. Obviously, the diagonal is the ideal case where the calculated probabilities coincide with the true probabilities. The fact that the curves lie above the diagonal is proof that the true value approach to EVS is conservative: for a given true compliance probability, the calculated one is always lower.

If the plot is made for a single contributor, all lines lie on the diagonal, as expected; if the number of contributors is increased, the two calculated curves move further away from the diagonal, meaning that the result becomes increasingly conservative.

The curve labeled COMPROB corresponds to an EVS calculation that does not utilize the true value approach, it is obtained with equation (8).

5. Conclusion

This paper presented an analytical expression that yields the same probabilities as the Monte Carlo approach to extreme value statistics based on true value estimates, applied to channel power compliance. This expression may be implemented in the calculations for ROP setpoints, which are based on analytical calculations rather than Monte Carlo, and may reduce required computing time.

6. References

[1] P. Sermer, C. Olive, and F. Hoppe, "Efficient Compliance with Prescribed Bounds on Operational Parameters by Means of Hypothesis Testing using Reactor Data", in Proceedings of the 21st Annual Canadian Nuclear Society Conference, Toronto, Ontario, Canada, June 11-14 2000.