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A Combined FEM-Ritz Method for Modeling 3D Acoustics in CANDU Fuel Sub-Channels

D. Villero and S. D. Yu
Department Of Mechanical and Industrial Engineering, Ryerson University,
Toronto, Canada
dvillero@ryerson.ca, syu@ryerson.ca

Abstract

The modeling and analysis of 3D acoustics in CANDU fuel sub-channels are conducted using a combined FEM-Ritz method to reduce the number of acoustic degrees of freedom for higher computational efficiency. The method uses a prism element by extruding an isoparametric six-node triangular element along the z-axis and computing the acoustic displacement by combining isoperimetric triangular shape functions and a polynomial shape function of n-th order. Lagrangian mechanics is utilized to assemble the equations of motion and derive all the system's matrices. The acoustic system consists of a cylindrical pressured pipe with rigid walls filled with fluid. At the inlet, an acoustic wave is generated by a time-harmonic source and, at the outlet, the acoustic wave interacts with a reacting and absorbing material. Numerical results obtained are found to be in excellent agreement with the analytic solution.

Keywords: Finite Element Method, Ritz Method, 3D Acoustics.

1. Introduction

Many engineering acousticproblems are modeled and solved usingcommercial software packages based on finite elementsthat offer adequate accuracy for an average computational time. However, when an acoustic system gets large or requires a fine mesh to capture local phenomena at a very detailed scale, the FE analysis becomes computationally expensive since the number of degrees of freedom may reach themillions.

In this paper, a combined FE-Ritz method is proposed and found suitable for handling low frequency acoustics in a geometrically complex 3DCANDU fuel channel filled with fuel bundles. The idea is to model 3D acoustic systems, where acoustic fluctuations on one plane can be captured by means ofFE analysis and the smooth acoustic fluctuations along the third axis can be approximated by polynomials of relative low orders. By employing this method, one is able to reduce the number of degrees of freedom of the desired problem and the computational time without jeopardizing the accuracy of the solution.

For the finite element formulation, a displacement-based method is proposed in order to examine the propagation of three-dimensional acoustic wavesina fluid medium. Compared to the classical pressure-based approach used by Craggs[1], Hackett[2], and Gladwell[3],a displacementbasedformulation by Yu and Kawall[4] allows the derivation of the equations of motion by Hamilton's variation principle, which facilitates the implementationof non-classical boundary conditions and interface conditions between distinct domains and media.

Thesix-node triangular isoparametric elements are used to mesh the geometrically complex cross-sectional plane; variations of acoustic displacement in the normal direction of the plane are handled using a polynomial interpolation. This method uses the polynomial expansion as a way of extruding the 2D mesh, so that the three-dimensional domain is divided into prism elements.As an illustrative example,Figure 1 shows a graphical representation of the domain discretization for the FE-Ritz method.The domain represents the fluid-filled region in a 36 rod fuel channel for a CANDU nuclear reactor.

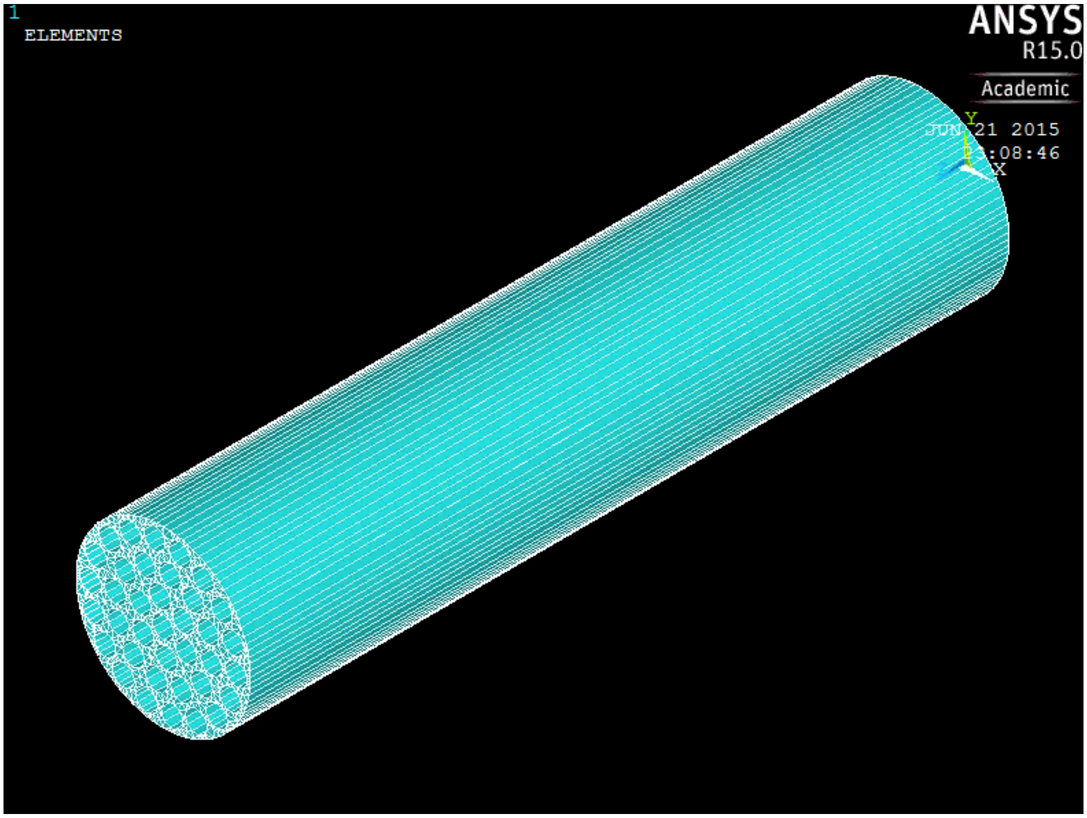

Figure 1:Three-dimensional system meshed with prism elements.

1.1 Assumptions

In this study, the following assumptions are made:1) the fluid flow is negligible; 2) the pipe walls are smooth, rigid and adiabatic; 3) fluid is inviscid; 4) the acoustic source can be modeled by a loud speaker at the inlet; 5) sound waves interact with a responsive and absorbing material at the outlet, hence an acoustic impedance is prescribed at the outlet. These assumptions are the same for every system model in this study. Essentially, these assumptions cause the acoustic waves to behave linearly and facilitate the validation of the results with the appropriate analytical solution for the system or by comparison against numerical results from a FE software such as ANSYS.

2. Background

The motivation for this work comes from the known bundle vibrationproblem in CANDU nuclear reactors.Under reactor operation, the bundles are known to vibrate. The sources of vibration are fluid flow and acoustic pressure pulsations originated at the main heat transport pumps and propagated to the fuel channels. Given the design of the bundles, they can show small-scale motions inside the tube such as rolling, sliding and bending vibration with respect to the designed equilibrium position. This motion leads to friction and impact between the bundles and the pressure pipe which, over a long period of time, appears to be damaging not only the pressure pipe but the fuel bundle as well. The bundle motion can cause wear and material-loss of the pressure pipe also known as fretting, along with endplate cracking.

In order to study this phenomenon, many numericaland experimental models have been developed(Bhattacharya[5], Paidoussis[6]; Zhang and Yu[7])to look at the effects of fluid-elastic instability, vibration induced by vortex shedding and turbulence-induced vibration. Other than experimental results such as those documented by Misra et al.[8],little work has been done in the area of computational acoustics, the main reason being the fact that constructing an acoustic model that can be easily coupled with a structural model is computationally challenging given the degrees of freedom of the system and the high resolution required to model the bundle system. The FE-Ritz method is adequate to solve this problem since it can reduce the degrees of freedom of the bundle system and it can be easily adapted to account for structure interaction due to its displacement based formulation.

3. Finite Element Formulation of Acoustic System

The acoustic system component is modeled using six-node isoparametricthree-dimensional prism elements, as seen in Figure 2.

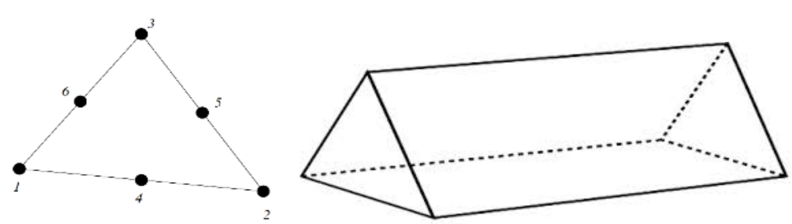

Figure 2: FE-Ritz three-dimensional prism element.

Within each element, the acoustic displacement varies with local coordinates, ξ, η and ζ,

$$u_e(x,y,z) = [N_1(\xi,\eta)]_{1X6} \begin{bmatrix} [N_2(\zeta)] & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & [N_2(\zeta)] \end{bmatrix}_{6X(6Xn)} \begin{Bmatrix} \begin{Bmatrix} \overline{u_0} \\ \vdots \\ \overline{u_n} \end{Bmatrix}_1 \\ \begin{Bmatrix} \overline{u_0} \\ \vdots \\ \overline{u_n} \end{Bmatrix}_2 \\ \vdots \\ \begin{Bmatrix} \overline{u_0} \\ \vdots \\ \overline{u_n} \end{Bmatrix}_6 \end{Bmatrix}_{(nX6)X1} \quad (1)$$

where,

$$[N_1(\xi,\eta,\gamma)]^T = \begin{bmatrix} \xi(2\xi-1) \\ \eta(2\eta-1) \\ (1-\xi-\eta)(2(1-\xi-\eta)-1) \\ 4(1-\xi-\eta)\eta \\ 4\eta(1-\xi-\eta) \\ 4(1-\xi-\eta)\xi \end{bmatrix} \quad (2)$$

is the shape function for a general six-node isoparametric triangle and $[\mathrm{N}_2(\zeta)] = [1 \;\; \zeta\, \zeta^2 \dots \zeta^{\mathrm{n}}]$ is the shape function for the linear combination of polynomials.

It is worth highlighting that the displacement along the y and z directions, $\mathrm{v}(\mathrm{x},\mathrm{y},\mathrm{z})$ and $\mathrm{w}(\mathrm{x},\mathrm{y},\mathrm{z})$ resembles the same form as that of $\bar{u}_e$. Also note that thefield variable chosen to formulate the equations of motion for the acoustic system is the acoustic displacement.

Moreover, the kinetic and potential energy can be written as:

$$T = \frac{1}{2}\iiint_{\Omega_e} \rho\, (\dot{\boldsymbol{u}})^2 dV = \sum_{e=1}^{N_e} \frac{1}{2}\{\dot{\bar{\boldsymbol{u}}}_e\}^T [M]_e \{\dot{\bar{\boldsymbol{u}}}_e\} \quad (3)$$

$$V = \frac{1}{2}\iiint_{\Omega_e} \rho c^2\, (\nabla \cdot \boldsymbol{u})^2 dV = \sum_{e=1}^{N_e} \frac{1}{2}\{\bar{\boldsymbol{u}}_e\}^T [K]_e \{\bar{\boldsymbol{u}}_e\} \quad (4)$$

Where $\{\bar{\boldsymbol{u}}_e\} = \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix}_e$, the element mass [M], and stiffness matrix [K] are calculated as:

$$[M_e] = \begin{bmatrix} [M]_e & 0 & 0 \\ 0 & [M]_e & 0 \\ 0 & 0 & [M]_e \end{bmatrix} \quad (5)$$

$$[M\ \ ]_e = \iiint_{\Omega_e} \rho [N_2(\xi,\eta)]^T [N_1(\zeta)]^T \, [N_1(\zeta)][N_2(\xi,\eta)]|J|\, d\xi\, d\eta\, d\zeta \quad (6)$$

And

$$[K]_e = \begin{bmatrix} [K_{xx}]_e & [K_{xy}]_e & [K_{xz}]_e \\ [K_{yx}]_e & [K_{yy}]_e & [K_{yz}]_e \\ [K_{zx}]_e & [K_{zy}]_e & [K_{zz}]_e \end{bmatrix} \tag{7}$$

Once the formulation for one element component vector is determined, one can relate the element displacement vector to the global component vector through a transformation matrix $[T_{e\to g}]$, as follows

$$\{\bar{\boldsymbol{u}}_e\} = [T_{e\to g}]\{\bar{\boldsymbol{u}}\} \tag{8}$$

One can further use this transformation to obtain the global component kinetic and potential energies as:

$$T = \frac{1}{2}\{\dot{\bar{\boldsymbol{u}}}\ \ \}^T[M]\{\dot{\bar{\boldsymbol{u}}}\ \ \} \tag{9}$$

$$V = \frac{1}{2}\{\boldsymbol{u}\ \ \}^T[K]\{\boldsymbol{u}\ \ \} \tag{10}$$

Where the component mass and stiffness matrix are:

$$[M] = \sum_{e=1}^{N_e}[T_{e\to g}]^T[M_e][T_{e\to g}] \tag{11}$$

$$[K] = \sum_{e=1}^{N_e}[T_{e\to g}]^T[K_e][T_{e\to g}] \tag{12}$$

3.1 Prescribed Acoustic Impedance

When acoustic impedance is imposed at one of the system's boundaries (i.e. the outlet of the pressure pipe), the boundary will dynamically affect the system as it responds to incoming acoustic energy. In a sense, this boundary absorbs and reflects some of the incoming acoustic wave which generates a pressure at all points on the boundary. This pressure can be interpreted as a generalized force $\{Q_p\}_e$, for a virtual displacement $\delta\{\bar{u}_e\}^T$, whose work done is defined as

$$W = -\sum_{e=1}^{N_e}\iint_\Gamma \delta \boldsymbol{W_e}^{\boldsymbol{T}} p\,dA \tag{13}$$

Where, Γ is the cross-sectional area at the boundary $\delta W_e{}^T = \delta\{u_e\}^T[N_2]^T[N_1]^T$ is the element virtual work and p is the generated pressure. The generalized forces are defined by the boundary's behavior or, in this case, the pressure generated by the specified acoustic impedance

$$\{p_e\} = k_a\{w_e\} + d_a\{\dot{w}_e\} \tag{14}$$

wherethe stiffness k_a and damping d_a coefficientscorrespond to the imaginary and real parts of the acoustic impendence,

$$\frac{\hat{p}_L}{\rho c \hat{v}_L} = Z_r + iZ_i \tag{15}$$

Where,$\hat{v}_L$ is the complex velocity at the boundary, Z_r is the real part of the acoustic impedance, Z_i is the imaginary part of the acoustic impedance,ρ is the density of the medium, c is the speed of the wave in the medium and ω is the frequency.

$$k_a = Z_i \rho c \omega \tag{16}$$

$$d_a = Z_r \rho c \tag{17}$$

Note that the pressure generated only depends on the axial displacement variation w. Therefore, the work done is calculated as:

$$W = -\sum_{e=1}^{N_e} \iint_\Gamma \delta\{\overline{w}_{\boldsymbol{e}}\}^T [N_2]^T [N_1]^T|_{z=L}(k_a[N_1]|_{z=L}[N_2]\{\overline{w}_e\} + d_a[N_1]|_{z=L}[N_2]\{\dot{\overline{w}}_e\})dA \tag{18}$$

And thus the generalized force can be written as

$$\{\boldsymbol{Q_p}\}_{e'} = \sum_{e=1}^{N_{e\Gamma}} \left(\delta\{\overline{w}_{\boldsymbol{e}}\}^T \ [\widetilde{K}_{zz}] \ \{\overline{w}_e\} + \delta\{\overline{w}_e\}[\tilde{C}\ \ _{zz}] \ \{\overline{\dot{w}\ \ _e}\}\right) \tag{19}$$

Note that the generalized force can be separated into two non-conservative damping and stiffness forces respectively, which are defined as:

$$[\tilde{C}\ \ ]_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & [\tilde{C}\ \ _{zz}]_e \end{bmatrix} \tag{20}$$

where

$$[\tilde{C}\ \ _{zz}]_e = \iint_\Gamma d_a[N_2]^T[N_1]^T|_{z=L}[N_1]|_{z=L}[N_2]|J|d\xi\, d\eta \tag{21}$$

And

$$[\widetilde{K}]_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{K}_{zz\,e} \end{bmatrix} \tag{22}$$

where

$$\left[\widetilde{K}_{zz}\right]_e = \iint_\Gamma k_a [N_2]^T [N_1]^T|_{z=L} [N_1]|_{z=L} [N_2]|J| d\xi\, d\eta \tag{23}$$

Since the generalized force only acts with respect to axial variations, the other components are zero. One can then add the stiffness contribution of the reacting boundary to the component stiffness matrix [K]. The global generalized forces can write as follows

$$\{\boldsymbol{Q_p}\} = -[\widetilde{K}\ \ ]\{\overline{\boldsymbol{u}}\ \ \} - [\tilde{C}\ \ ]\{\dot{\overline{\boldsymbol{u}}}\ \ \} \tag{24}$$

For which the component non-conservative stiffness and damping matrices are

$$[\widetilde{K}\ \ ] = \sum_{e=1}^{N_e} \left[T_{e\to g}\right]^T [\widetilde{K}\ \ ]_e \left[T_{e\to g}\right] \tag{25}$$

$$[\tilde{C}\ \ ] = \sum_{e=1}^{N_e} \left[T_{e\to g}\right]^T [\tilde{C}\ \ ]_e \left[T_{e\to g}\right] \tag{26}$$

3.2 Equations of Waves

The equations of waves in the fuel string acoustic system may be obtained from the following Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\{\dot{\overline{\boldsymbol{u}}}\ \ \}}\right) + \frac{\partial L}{\partial\{\overline{\boldsymbol{u}}\ \ \}} = \{\boldsymbol{Q}\ \ \} \tag{27}$$

Substituting Equations(9), (10) and (24) into Equation (27) yields the following equations of motion for the unconstrained system

$$[M]\{\ddot{\overline{\boldsymbol{u}}}\ \ \} + [K]\{\overline{\boldsymbol{u}}\ \ \} = -[\widetilde{K}\ \ ]\{\overline{\boldsymbol{u}}\ \ \} - [\tilde{C}\ \ ]\{\dot{\overline{\boldsymbol{u}}}\ \ \} \tag{28}$$

4. Plane Wave Solution

In order to test the previous FE formulation, one can manipulate the boundary conditions to model simple systems whose solution is already known. Ideally, if sound were to travel through a pipe with smooth and rigid walls, its behavior would be that of a plane wave. Hence, imposing these conditions to the proposed model should yield the same results as those from a plane wave case. Sinceit is assumed that, at the outlet of the system, the wave will interact with a reacting and absorbing wall, the pressure along the pipe is given byYu[4]as

$$\hat{p}_L(z) = \rho c V_0 \frac{(Z_r + iZ_i)\cos k(L-z) - i\sin k(L-z)}{\cos kL - i(Z_r + iZ_i)\sin kL} \tag{29}$$

Eq. (29)is the analytical solution for the acoustic complex pressure everywhere if the walls of the pipe are smooth and rigid. For this test case, it is not necessary to completely discretize an entire domain, but rather use one element as depicted in Figure 2, which essentially models a prismatic pipe.

5. Results

This section presents the results obtained from the plane wave test case within one prism element and the results for a global system that simulates the propagation of acoustic waves throughout a one-sixth sector of the cylindrical domain depicted in Figure 1.

5.1 Plane Wave Results

Numerical results are compared against the analytical solution in both space and time domain, where the space domain illustrates the steady state solution of the system after iteration time and the time domain illustrates the convergence time for the acoustic response.

Within one FE-Ritz element, the plane wave results are calculated using the followingparameters:

Table 1: Values of Geometric Properties Used for Plane Wave Test Case in Air.

Parameters	
Density $\rho\left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)$	1.2
Length L (m)	1.705 m
Speed of sound $c(\frac{m}{s})$	341 m/s
Piston velocity V_0 $(\frac{m}{s})$	0.01 m/s
Real acoustic impedance Z_R	4
Imaginary acoustic impedance Z_I	3
Excitation frequency ω $(\frac{\mathrm{rad}}{s})$	270
Time step	0.001 sec
Total time	0.2 sec
Order of polynomial n	5

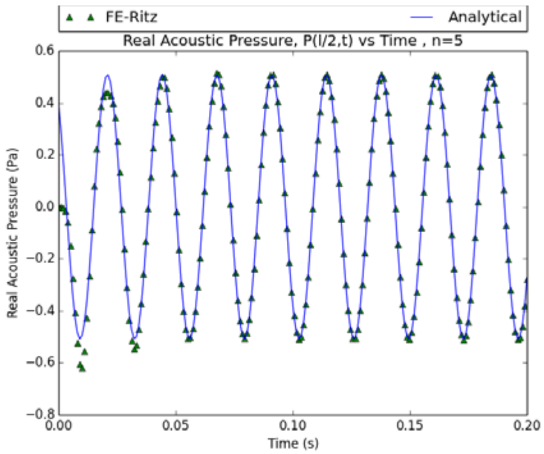

Figure 3: Time domain convergence of FE-Ritz method for plane wave test case in air.

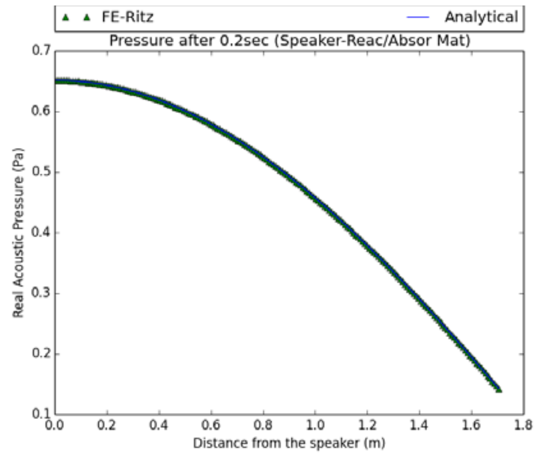

Figure 4: Steady state acoustic pressure for plane wave test case in air.

In Figures 3 and 4, it can be seen that both the analytical and numerical solution are in excellent agreement.

The time domain results inFigure 3 and Figure 5 show that the computed solution converges within the first three oscillations; hence the computed responses are phase-shifted in order to numerically compare the results against the analytical solution. The acoustic impedance for both water and air are arbitrarily calculated based on the media in which sound propagates and then stiffness and damping characteristics of the material at the outlet.

5.2 3D Acoustic System Results

The following results are computed using the parameters fromTable 2,for the assumptions stated in the introductory sectionand in afraction of the domain depicted inFigure 1.The numerical results are then compared against numerical results obtained from ANSYS for a model with the same characteristics.

Table 2: Values of Geometric Properties in water.

Parameters	
Density $\rho \left(\frac{kg}{m^3}\right)$	1000
Length L (m)	0.5 m
Radius r (m)	0.06
Piston pressure on inlet areaP_0 $\left(\frac{Kgm}{s^2}\right)$	$5\ \frac{Kgm}{s^2}$
Real acoustic impedance Z_R	4
Imaginary acoustic impedance Z_I	0

Excitation frequency ω ($\frac{\text{rad}}{\text{s}}$)	500
Time step	0.00001 sec
Total time	0.1 sec
Order of polynomial n	5

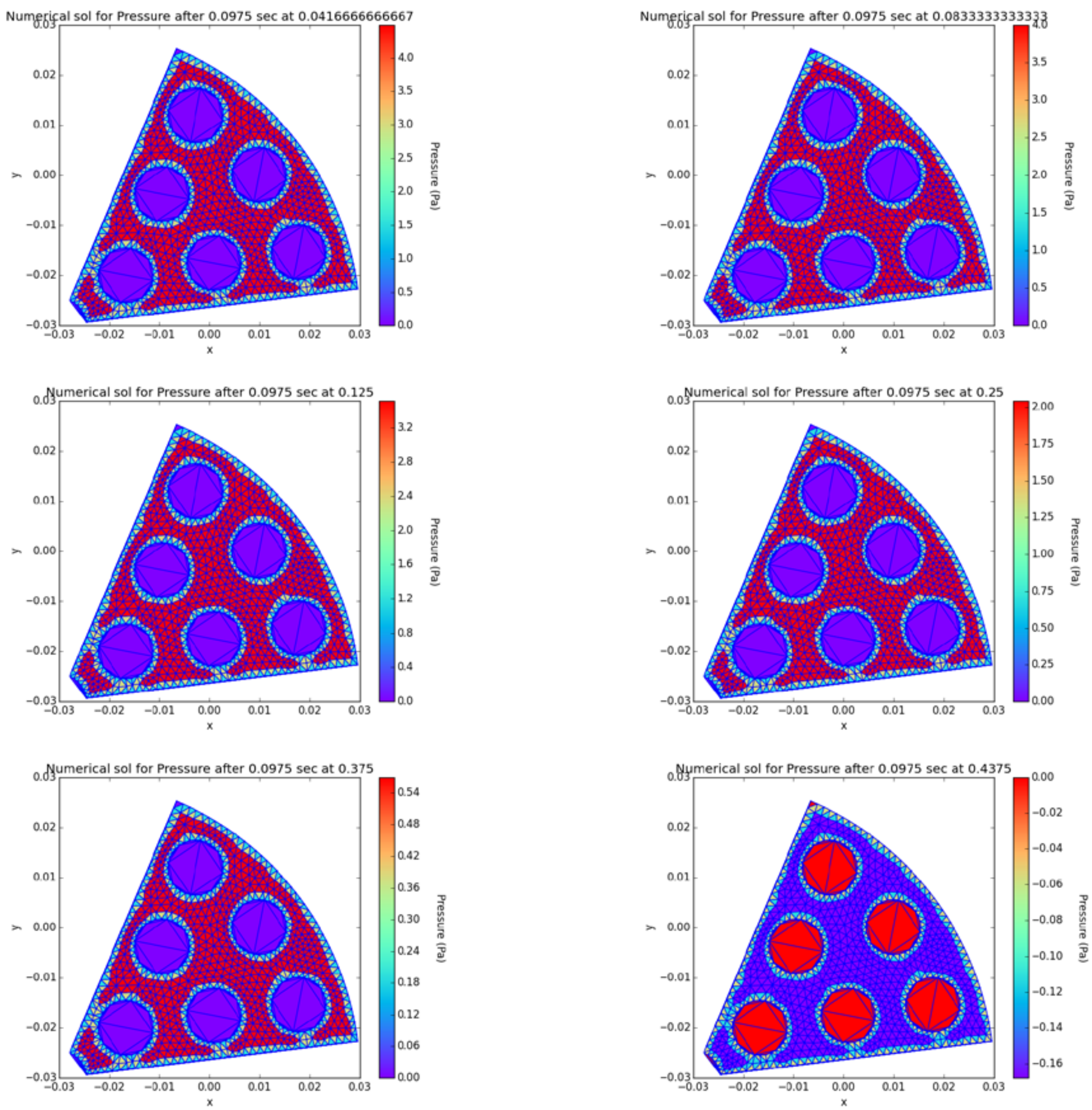

Figure 5 Steady state cross-section pressure at different z locations: L/12, L/8, L/4,L/2, 3L/4 and 7L/8.

For simplicity, the boundaries have been assumed to be smooth and rigid. Note that the pressure distribution on each cross-section changes along the length of the pipe. Thus, one can see that the pressure decreases smoothly from the inlet to the outlet.

Using ANSYS to compute the pressure after 0.1 seconds with the same characteristics and the same incident pressure generates the following pressure distribution:

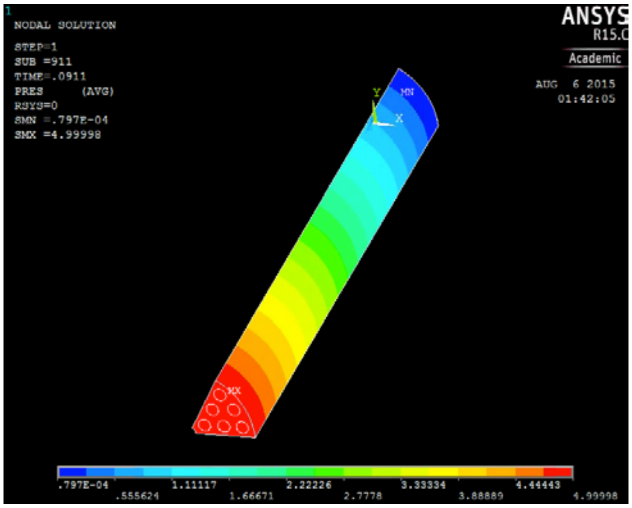

Figure 6: Steady state for pressure distribution along sector (ANSYS solution).

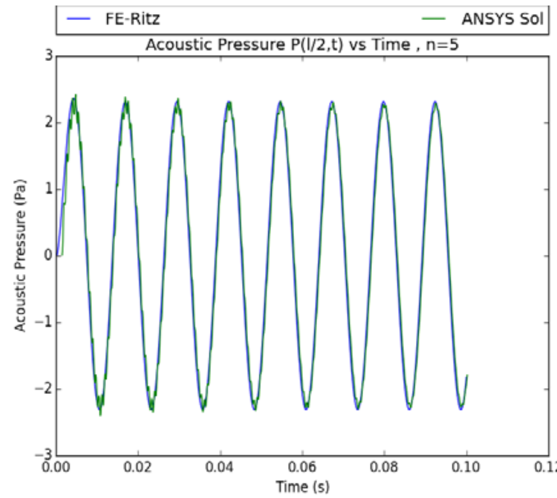

Figure 7: Time domain response for one node comparison (FE-Ritz vs. ANSYS).

The results depicted in Figure 5 are within the same range of values at the corresponding cross-section when compared to the results in Figure 6. Further, the results in Figure 7 appear to be shifted by a face angle. This is because the results from ANSYS converge after 4 cycles due to the low order element (8-node bricks) used while the FE-Ritz method achieves convergence within the first 2.

6. Conclusion

In this study, the acoustic behavior of fluid particles inside a pressure pipe is investigated using a combination Finite Element method and Ritz method. The theoretical procedure presented in this paper has been implemented into a Fortran 77 code and a Python code in order to facilitate the use for a wide range of applications including the bundle vibration problem of the CANDU system. Also, the code can be easily modified in order to study three dimensional acoustic-structure interactions with the help of an automated mesh generator or by using stating meshing, which may only apply for low frequency excitations. Additionally, the analysis done by the code is faster than the analysis done by a regular 3D finite element module, simply because the FE-Ritz method reduces the degrees of freedom and therefore requires less memory and CPU. The plane wave test case and the sector model haves shown to accurately match the analytic solution obtained from Yu[4] and the ANSYS numerical results, which validates the method and its implementation.

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