

BUBBLE DYNAMICS IN SINGLE COMPONENT FLUID

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Abstract

The response of bubbles-cloud in thermodynamically meta-stable liquid influences all acoustic processes in multiphase flows. System computer codes with single pressure concept widely used in the industry does not take this into account. This paper demonstrates the need of extension of the existing computer codes to at least two-pressure concepts in order to adequately describe for instance condensation oscillations of steam in sub-cooled liquid. Review of the state of the art is given in Appendixes. The symmetric bubble dynamics mathematical description is proposed based on “two-phase” Rayleigh-Plesset equation together with analytical model for the bubble parameters and high accuracy numerical solution of the heat conduction problem. The model is validated on evaporation and condensation experiments. Finally self-triggered oscillation system is demonstrated for 5 mm bubbles in strong sub-cooled water resulting in high internal bubble pressure

1. INTRODUCTION

If liquid is heated for whatever reason at temperatures higher than the saturation temperature at the local pressure generates bubbles. Vapour enclosed in cavity with interface colder than the saturation temperature at the bubble inside pressure condenses and causes bubble-collapse. In principle, the pressure difference between the inside and outside of a bubble cause the mechanical movement resisted by the viscosity and surface tension. Simultaneously, heat and mass transfer at both sites of the interface cause pressure change and then again this pressure change has a feed back on bubble interface motion. After more than 80 years of research we finally know the phenomena and the equations describing it. The literature describing bubble radius as a function of initial parameters and time by solving these equations with different degree of complexity is huge, mainly because of the attempts to obtain closed analytical solutions for which obtaining simplifications are inevitable. But now the computers are so powerful that direct numerical simulation of the one-dimensional problem is easily quickly done. The subject of this paper is to demonstrate such a solution, to relate it to the state of the art and to compare it with experimental data. Finally, a discussion is provided how to couple such on small scale physics with large scale physics in multi-phase flow computer codes.

2. THE SYSTEM OF PDE'S DESCRIBING THE PROBLEM

Consider a symmetric bubble in incompressible liquid. The equations describing the process are:

$$R_1^2 \frac{d^2 R_1}{d\tau^2} + \frac{3}{2} R_1 \left(\frac{dR_1}{d\tau} \right)^2 + 4\eta_2 \frac{dR_1}{d\tau} + \frac{1}{\varepsilon \rho_2} [2\sigma_2 + R_1 (p_2 - p_1)] = 0, \quad (1)$$

$$\frac{\partial T_2}{\partial \tau} - a_2 \frac{\partial}{\partial x} \left(\frac{\partial T_2}{\partial x} \right) - 2 \frac{a_2}{r} \frac{\partial T_2}{\partial x} = 0, \quad x = r - R_1(\tau), \quad (2)$$

$$p_1 \approx p_{10} \left(\frac{R_{10}}{R_1} \right)^{3\kappa} \times \exp \left\{ \frac{3\kappa \Delta \tau}{R_1} \left[- \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) (\rho w)_2^{1\sigma} + \frac{\dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} (T'(p_1) - T_1)}{\rho_1 c_{p1} T_1} \right] \right\} \quad (3)$$

$$T_1 \approx T_{10} \left(\frac{p_1}{p_0} \right)^{(\kappa-1)/\kappa} \times \exp \left\{ \frac{\kappa-1}{\kappa} \frac{3\Delta \tau}{R_1 p_1} \left[\dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} (T'(p_1) - T_1) \right] \right\}, \quad (4)$$

$$\frac{dR_1}{d\tau} = \left(\frac{dR_1}{d\tau} \right)_{mom} - \frac{1}{\rho_2} (\rho w)_2^{1\sigma}, \quad (5)$$

see [18, 42].

Equation 1 is the momentum equation for an incompressible liquid integrated from the radius of the bubble to infinity. This equation is known as the Rayleigh-Plesset equation. Appendix 1 gives its derivation.

Equation 2 is the energy conservation equation for a symmetric incompressible liquid in spherical coordinates. The origin of the coordinate system is at the inter-face of the bubble. The spatial coordinate transformation used to eliminate the convective term is $x = r - R(\tau)$, see Appendix 2.

Equation 3 is the so called pressure equation or volume conservation equation for the bubble. Its derivation is given in Appendix 3.

Equation 4 is the energy conservation equation in pseudo-entropy form. The derivation is given in Appendix 4.

The change of the bubble radius with the time consists of the mechanical component $(dR_1/d\tau)_{mom} = (R_1^{n+1} - R_1)/\Delta \tau$ dictated by the force balance (Eq. 1) and by a component depending on the mass transfer as given with Equation 5.

The interfacial heat fluxes are defined as follows. The heat flux from the inter-face to the vapor is defined by Eq. 6. The heat flux from the interface to the liquid is defined by Eq. 7. The interfacial energy jump condition is expressed by Eq. 8.

$$\dot{q}_1^{n2\sigma} = htc_1^{2\sigma} (T_1^{2\sigma} - T_1), \quad (6)$$

$$\dot{q}_2^{n1\sigma} = \lambda_2 \left(\frac{\partial T_2}{\partial x} \right)_{x=0} = htc_2^{1\sigma} (T_2^{1\sigma} - T_2), \quad (7)$$

$$(\rho w)_2^{1\sigma} = \frac{htc_2^{1\sigma} (T_2^{1\sigma} - T_2) + htc_1^{2\sigma} (T_1^{2\sigma} - T_1)}{h_1^{2\sigma} - h_2^{1\sigma}}. \quad (8)$$

The gas and liquid side heat transfer coefficients are formally $htc_1^{2\sigma}$ and $htc_2^{1\sigma}$. Some authors prefer to use the surface temperature as independent variable by enforcing a kinetic condensation and evaporation at the surface using the so called accommodation coefficients. The reader will find such examples in [32] and [24]. The gradient of the liquid temperature field needed in Eq. 7

results from the solution of the liquid energy conservation equation. The liquid side heat transfer is essential for single component systems. The gas side heat transfer is important for gas bubbles and pressure oscillations not allowing mass transfer. The gas side heat transfer coefficient can be computed either by solving the energy conservation equation for the gas or by means of a not very accurate empirical relations in which

$$Nu_1^{2\sigma} = \frac{h c_1^{2\sigma} 2R_1}{\lambda_1} \quad (9)$$

where $Nu_1^{2\sigma} = 1$ for stagnant bubble. For moving bubble $|\Delta V_{12}| > 0$, which is not the subject of this consideration, due to the internal circulation the heat transfer improvement is a function of the relative velocity. [6] correlated their experimental data for condensation on droplets with the following expression for the *Nusselt* number,

$$Nu_1^{2\sigma} = \max(1, 0.53 Pe_1^{*0.454}) \quad (10)$$

where the definition of the *Peclet* number is used

$$Pe_1^* = \frac{D_3 |\Delta V_{13}|}{a_1} \frac{\eta_1}{\eta_1 + \eta_3}. \quad (11)$$

For bubble interaction one can postulate

$$Pe_1^* = \frac{D_1 |\Delta V_{12}|}{a_1} \frac{\eta_2}{\eta_1 + \eta_2} \quad (12)$$

instead. Whether the coefficient will remain the same also for bubble internal circulation is not clear. *Nigmatulin* [23] derived approximate formula to compute the gas-side heat transfer due to surface motion of stagnant bubble:

$$Nu_1^{2\sigma} = \sqrt{Pe} \text{ for } Pe > 100, \quad (13)$$

$$Nu_1^{2\sigma} = 10 \text{ for } Pe \leq 100, \quad (14)$$

where

$$Pe = 12(\kappa - 1) \frac{T_2}{|T_2 - T_1|} \frac{R_1}{a_1} \frac{dR_1}{d\tau} = 6(\kappa - 1) \frac{T_2}{|T_2 - T_1|} \frac{\rho_1 2R_1}{\eta_1} \frac{dR_1}{d\tau} \frac{\eta_1 c_{p1}}{\lambda_1}. \quad (15)$$

3. NUMERICAL SOLUTION METHOD

The *Rayleigh* equation is discretized by using a second order method

$$R_1^2 \frac{R_1^{n+1} - 2R_1 + R_1^{n-1}}{\Delta\tau^2} + \frac{3}{2} R_1 \left(\frac{R_1^{n+1} - R_1^{n-1}}{2\Delta\tau} \right)^2 + 4\eta_2 \frac{R_1^{n+1} - R_1^{n-1}}{2\Delta\tau} + \frac{1}{\varepsilon\rho_2} [2\sigma_2 + R_1 (p - p_1)] = 0 \quad (16)$$

or

$$a(R_1^{n+1})^2 + bR_1^{n+1} + c = 0 \quad (17)$$

where,

$$a = \frac{3}{8} R_1,$$

$$b = R_1^2 + 2\eta_2 \Delta\tau - 2aR_1^{n-1},$$

$$c = (R_1^2 - 2\eta_2 \Delta\tau) R_1^{n-1} - 2R_1^3 + a(R_1^{n-1})^2 + \frac{1}{\varepsilon\rho_2} [2\sigma_2 + (p - p_1) R_1] \Delta\tau^2$$

with the solution

$$R_1^{n+1} = \left(-b + \sqrt{b^2 - 4ac} \right) / (2a). \quad (18)$$

The liquid energy conservation equation is solved using a second order method of *Lax* and *Vendroff*, see in [30], applied as follows. First, an explicit half time step is made and all temperatures are computed

$$T_{2,i}^{n+1/2} - T_{2,i} - \frac{1}{2} \frac{a_2 \Delta\tau}{\Delta x^2} \left[T_{2,i+1} - 2T_{2,i} + T_{2,i-1} + \frac{\Delta x}{r_i} (T_{2,i+1} - T_{2,i-1}) \right] = 0. \quad (19)$$

With the new temperature values the spatial terms are computed again and a complete time step is then made

$$T_{2,i}^{n+1} - T_{2,i} - \frac{a_2 \Delta\tau}{\Delta x^2} \left[T_{2,i+1}^{n+1/2} - 2T_{2,i}^{n+1/2} + T_{2,i-1}^{n+1/2} + \frac{\Delta x}{r_i} (T_{2,i+1}^{n+1/2} - T_{2,i-1}^{n+1/2}) \right] = 0. \quad (20)$$

The dimensionless time $a_2 \Delta\tau / \Delta x^2$ is called grid *Fourier* number. $\Delta x / r_i$ indicates the importance of the sphericity close to the bubble. The ordinary partial differential equations 3 and 4 are solved by first order *Euler* method giving the new pressure p_1 and temperatures T_1 inside the bubbles in both steps. Then the bubble side interface temperature is the saturation temperature at the vapour pressure p_1 and at the other end no heat flux is specified: $T_{2,1}^{n+1/2} = T'(p_1)$ and $T_{2,1}^{n+1} = T'(p_1)$ and $T_{2,i_{\max}}^{n+1/2} = T_{2,i_{\max}-1}^{n+1/2}$ and $T_{2,i_{\max}}^{n+1} = T_{2,i_{\max}-1}^{n+1}$. The x -region along which the liquid energy conservation equation is integrated is associated with the density of the bubbles per cubic meter n_1 as follows

$$\sum \Delta x_i = \left(\frac{3}{4\pi n_1} \right)^{1/3} - R_{i0} = R_{i0} (\alpha_1^{-1/3} - 1). \quad (21)$$

Actually *Nigmatulin* proposed in 1978 the so called concept of “cellular medium” considering the processes within the sphere with $R_{cell} = R_{i0} \alpha_1^{-1/3}$. This allows taking into account the effective thickness of liquid layer around each bubble. The larger the bubble concentration at a given initial radius is, the smaller the thickness of the effective layer. This leads to different behaviour of the bubbles depending on the bubble number density. Note the difference to [19] using

$$\sum \Delta x_i = 0.002 \text{ m}$$

which does not couple the problem to the multiphase flows.

4. VALIDATION OF THE METHOD

4.1 Bubble collapse

I will first validate the method by using the bubble collapse experiments by [7] for water. Given was the initial pressure, subcooling and bubble radius, see Table 1.

Table 1. Initial and boundary conditions for the experiment by [7]

Figs. in [7]	p in bar	R_{i0} in mm	$T' - T_2$ in K
9	1	3.657580	12.2
10	1	3.185503	8.7
11	1	2.983309	8.3
12	1	4.372804	5.6
13	1	3.990081	5.0

Measured was the bubble radius as a function of time. This is also predicted by the method and compared in Figs. 1 to 5. First of all we observe from Fig. 1 that there are residual non-condensable gases in the bubble because otherwise the stable final diameter of 0.25 mm can not be explained. Having this in mind, the very good agreement with the initial bubble collapse period and the complete condensation for a single component system is well understood. The most remarkable feature is that the selected numerical method does not damp the natural oscillations contained in the Rayleigh equation. The inertia leads to overshooting. The surface tension acts like a spring. The viscosity introduces little damping. The accuracy of the integration influences the resolution of the pressure amplitudes as seeing from Fig. 1. The experiment on Fig. 2 also indicates residual gases sufficient to stop the condensation at 0.5 mm. Similar is the situation with the experiment presented on Fig. 3 where the final radius is about 0.4 mm.

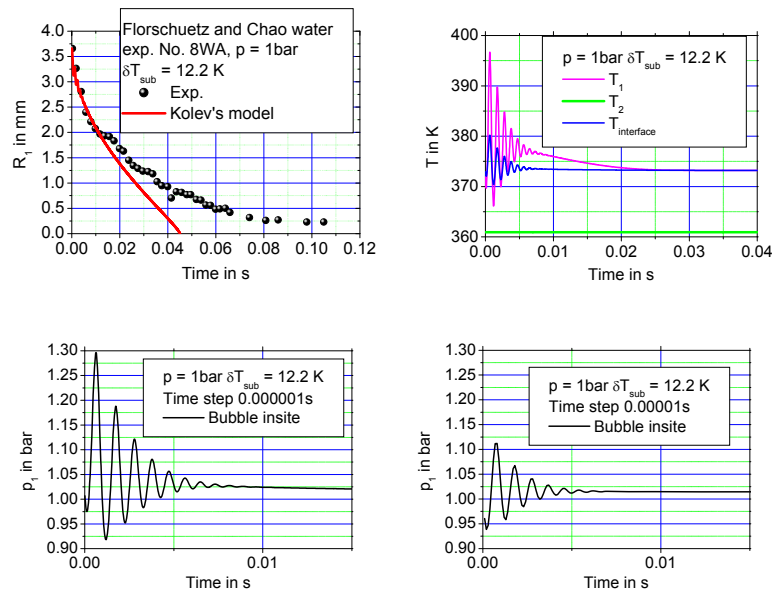


Fig. 1. Florschuetz and Chao [7] experiment for water: a) Bubble radius as function of time; b) Bubble interface temperature as a functions of time; c) Pressure inside the bubble as a function of time, time step of integration 0.001 ms; time; d) Pressure inside the bubble as a function of time, time step of integration 0.01 ms

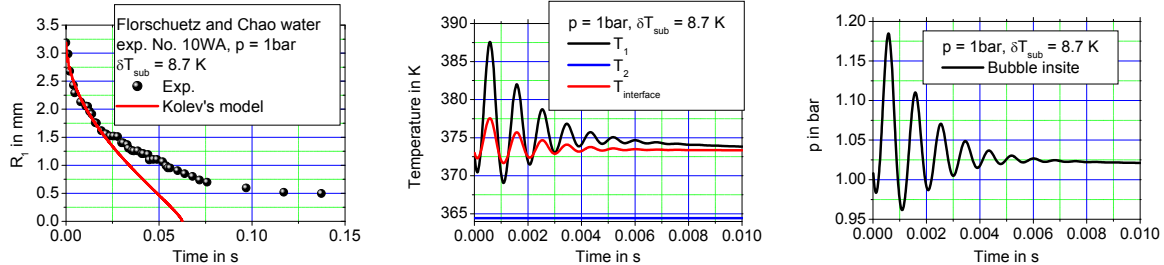


Fig. 2. Florschuetz and Chao [7] experiment for water: a) Bubble radius as function of time; b) Bubble interface temperature as a functions of time; c) Pressure inside the bubble as a function of time, time step of integration 0.01 ms

Figures 5 and 6 indicate excellent agreement with the data.

Conclusion: The so formulated numerical method is in a good position to adequately describe bubble collapse without limiting simplifying assumptions characteristic for closed analytical solutions.

Now let us see the influence of the increasing subcooling on the collapse process. Fig. 6 presents a computed collapse of initially 5mm-bubble in water at atmospheric pressure with subcooling from 5 to 80 K. In small subcoolings there are only slight oscillations of the bubble radius. The amplitudes increase with the increasing subcooling.

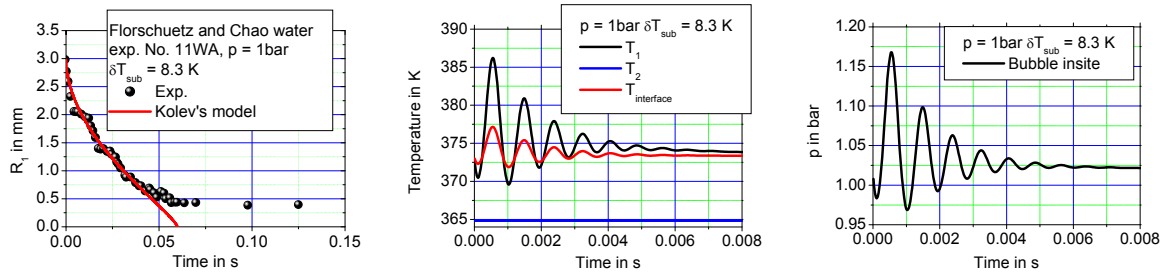


Fig. 3. Florschuetz and Chao [7] experiment for water: a) Bubble radius as function of time; b) Bubble interface temperature as a functions of time; c) Pressure inside the bubble as a function of time, time step of integration 0.01 ms

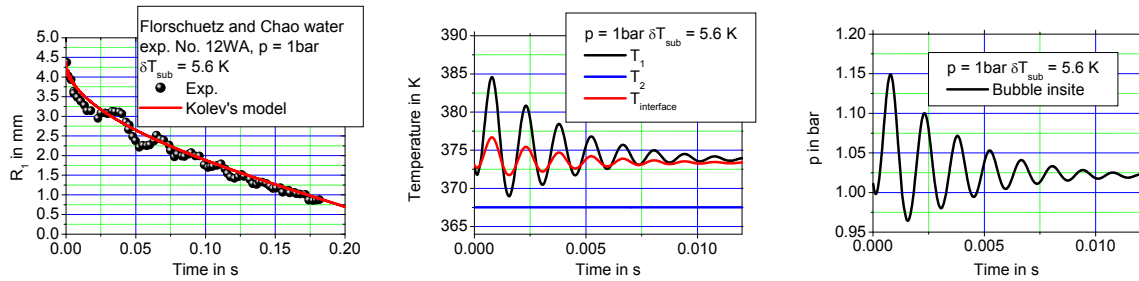


Fig. 4. Florschuetz and Chao [7] experiment for water: a) Bubble radius as function of time; b) Bubble and interface temperature as a functions of time; c) Pressure inside the bubble as a function of time, time step of integration 0.01 ms

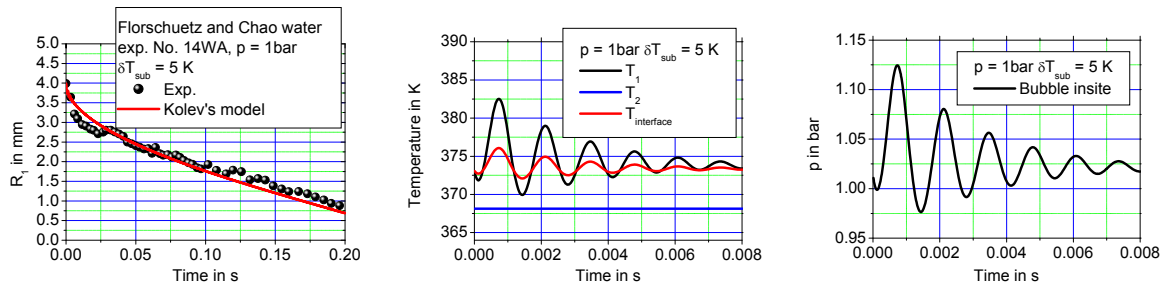


Fig. 5. Florschuetz and Chao [7] experiment for water: a) Bubble radius as function of time; b) Bubble and interface temperature as a functions of time; c) Pressure inside the bubble as a function of time, time step of integration 0.01 ms

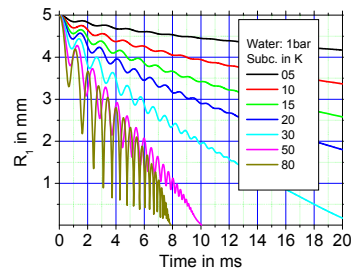


Fig. 6. Computed bubble radius as function of time for water at 1 bar and different subcoolings. Void 0.1

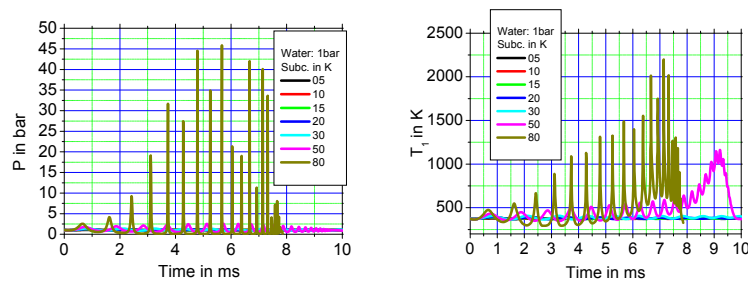


Fig. 7. Computed bubble pressures and temperatures as functions of time for water at 1 bar and different subcoolings. Void 0.1

Figures 7 show the corresponding pressures and temperatures inside the bubble. At 80 K subcooling we see strong pressure and temperature amplitudes. This explains the observed triggering of steam explosions in melt-water mixtures in which the water is strongly subcooled. The collapsing bubbles generate pressure waves which may lead to such a local pressure pulse generation that can trigger thermal explosion of a molten drop falling in water.

Conclusions: The presented numerical method does not damp the strong bubble oscillations in strong subcooling which is very remarkable.

4.2 Bubble growth

The experimental data by *Plesset* and *Zwick* [24] presents the bubble size as a function of time for sub-atmospheric pressure. The bubbles are generated on heated surface at sub-atmospheric pressure and wall superheat given in Table 2.

Table 2. Bubble growth experiments on heated plate performed by Plesset and Zwick [24]

p in Pa	T' in K H_2O/Na	$T_w - T'$ in K	Ja H_2O/Na
101325	373.12/1154.6	3.1	9.3/2.78
101325	373.12/1154.6	4.5	13.48/4.04
101325	373.12/1154.6	5.3	15.9/4.76

The initial size in the computations was set to the critical size computed as follows $R_{lc} = [2\sigma_2 T''(p)] / (\rho'' \Delta h \Delta T_{sup})$. The predicted sizes are presented in Figs. 8, 9 and 10.

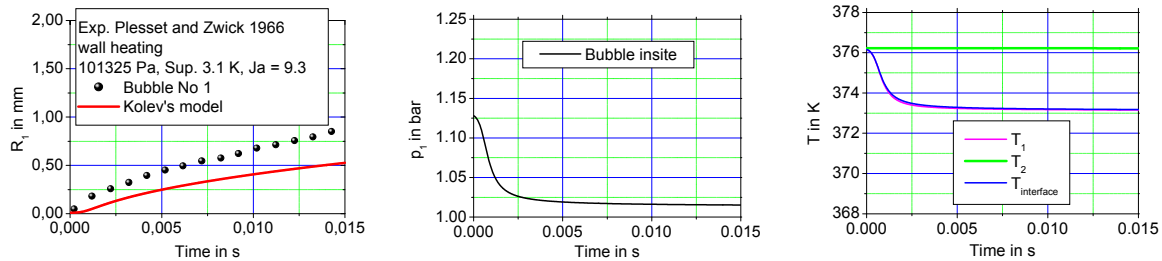


Fig. 8. Bubble diameter as function of time. Bubble generated at heated wall. Experimental data for water by Plesset and Zwick [24]. Comparison with the theoretical prediction

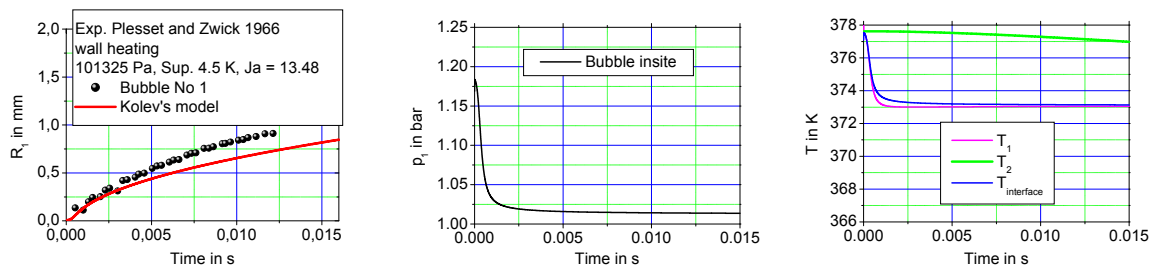


Fig. 9. Bubble diameter as function of time. Bubble generated at heated wall. Experimental data for water by Plesset and Zwick [24]. Comparison with the theoretical prediction

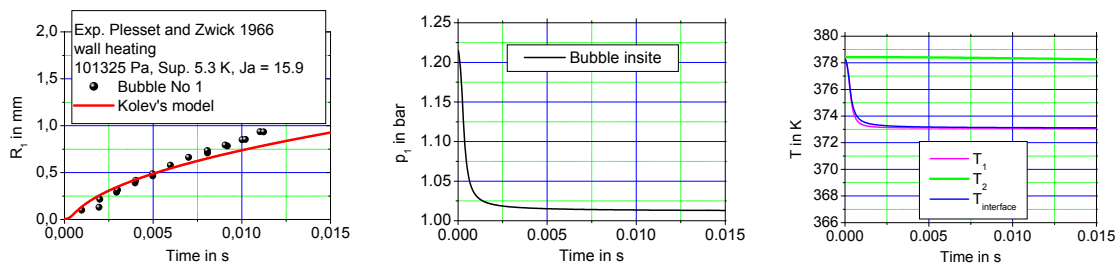


Fig. 10. Bubble diameter as function of time. Bubble generated at heated wall. Experimental data for water by Plesset and Zwick [24]. Comparison with the theoretical prediction

The comparisons show that the model predicts the initial bubble growth period well and then starts to deviate at higher bubble sizes – probably because the bubbles detach and the relative motion not taken into account here improve the interfacial heat transfer. Note, that the evaporation is a much more stable process compared to the condensation.

4.3 USE IN COMPUTER CODES OPERATING WITH LARGE COMPUTATIONAL STEPS

This author does not know a system computer code simulating multiphase flows by using more than one pressure in a cell. Although 30 years ago it was found that two pressure systems recover hyperbolicity etc. this still did not find application in modern code design. Now if we are about to resolve acoustic processes the question arises: is it possible with single pressure models? “The answer is no, it is not.” See the pressure inside the bubble from Figures 1 through 5. Obviously, only a separate pressure field for the vapour can have the memory along the time axis at which moment the mono-disperse bubble system in a cell is being during its dynamics. What is possible now is to trace the processes in and around the bubble with a much smaller time step, to compute the integral heat and mass transfer terms and to supply them to the macro-scale solver. An example for a mixture with 0.1 volumetric part void is given in figures 11 a) and b). Figure 11 a) shows the local instantaneous mass source term and figure 11 b) the mass source term averaged over the elapsed time since the beginning of the process. Using the second term in a system computer code using locally time averaged equations is the correct way to reach ultimate stability. But the accuracy control remains a question of proper selection of the time step.

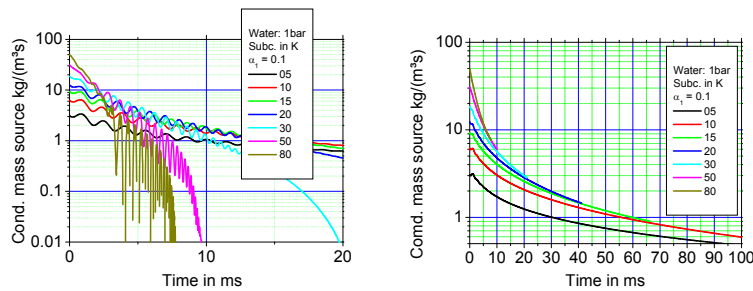


Fig. 11. Condensed mass per unit mixture volume and unit time as a function of the time for 0.1 volume concentration saturated void with initial bubble size 5mm. Liquid pressure 1bar. Parameter: water subcooling. a) Instantaneous mass source; b) Averaged over the elapsed time

In any case I should emphasize once again that without the second pressure for the vapor an important acoustic part of the processes can not be simulated in system computer codes. Such is the case for the well known dependence of the travel velocity of oscillations that is depending on the frequencies.

5. COMPARISON WITH THE STATE OF THE ART

The best review to this subject is given by [21] and I will not repeat it here. *Rayleigh* [28] considered the zero-pressure cavity collapse controlled only by the equality of the compression energy and the kinetic energy of the liquid resulting in

$$\frac{3}{2} \left(\frac{dR_1}{d\tau} \right)^2 + \frac{p}{\rho_2} \left(\frac{R_{10}^3}{R_1^3} - 1 \right) = 0 \quad (22)$$

and after integration computed the time necessary for the complete collapse

$$\Delta \tau_{col} = R_{10} \sqrt{\frac{2\rho_2}{3p}} \int_{\beta}^1 \frac{\beta^{3/2}}{(1-\beta^3)^{1/2}} d\beta \approx 0.9168 R_{10} \sqrt{\frac{\rho_2}{p}}, \quad (23)$$

where $\beta = (R_1/R_{10})^3$. Then *Rayleigh* evaluated the pressure field around the bubble during the bubble collapse by integrating the mass and momentum equation of the liquid

$$u_2 = \frac{dR_1}{d\tau} \left(\frac{R_1}{r} \right)^2, \quad (24)$$

$$\frac{\partial u_2}{\partial \tau} + u_2 \frac{\partial u_2}{\partial r} - \frac{1}{\rho_2} \frac{\partial p_2}{\partial r} = 0. \quad (25)$$

So the main direction for the description of the mechanical interaction in the incompressible liquid was given by *Rayleigh* and remained up to now. Equation 25 is equation 1 with neglected viscosity surface tension and set bubble pressure equal to zero. Therefore the result by *Rayleigh* is already contained in our system of PDEs.

Plesset and *Zwick* [24] introduced the liquid energy equation for incompressible liquid. They found the following solution for the surface temperature

$$T_2^{1\sigma}(\tau) - T_2 = -\sqrt{\frac{a_2}{\pi}} \int_0^\tau \frac{R_1^2(\tau^*) \frac{dT_2^{1\sigma}}{d\tau}}{\left[\int_{\tau^*}^\tau R_1^4(\tau^{**}) d\tau^{**} \right]^{1/2}} d\tau^* \quad (26)$$

which, for very small radius changes, reduces to

$$T_2^{1\sigma}(\tau) - T_2 = -\sqrt{\frac{a_2}{\pi}} \int_0^\tau \frac{\frac{dT_2^{1\sigma}}{d\tau}}{(\tau - \tau^*)^{1/2}} d\tau^*. \quad (27)$$

The combination of momentum and energy equation is at this point still not used. *Plesset* and *Zwick* [24] assumed no difference between the surface and vapor temperature and therefore

$$(\rho w)_2^{1\sigma} = \frac{\lambda_2}{\Delta h} \left(\frac{\partial T_2}{\partial x} \right)_{x=0}. \quad (28)$$

In this case the vapor mass conservation

$$\frac{d}{d\tau} \left(\rho_1 \frac{4}{3} \pi R_1^3 \right) = -4\pi R_1^2 (\rho w)_2^{1\sigma} \quad (29)$$

gives the relation between bubble radius and temperature gradient. Neglecting the compressibility of the vapour the authors used

$$\left(\frac{\partial T_2}{\partial x} \right)_{x=0} = - \frac{\rho_1 \Delta h}{\lambda_2} \frac{dR_1}{d\tau} \quad (30)$$

in Eq. 26 resulting in

$$T_2^{1\sigma}(\tau) - T_2 = \frac{\rho_1 \Delta h}{\lambda_2} \sqrt{\frac{a_2}{\pi}} \int_0^\tau \frac{R_1^2(\tau^*) \left(\frac{dR_1}{d\tau} \right)_{\tau^*}}{\left[\int_{\tau^*}^\tau R_1^4(\tau^{**}) d\tau^{**} \right]^{1/2}} d\tau^* . \quad (31)$$

assuming that $p_1 = p'(T_2^{1\sigma})$ in Eq. 1,

$$R_1^2 \frac{d^2 R_1}{d\tau^2} + \frac{3}{2} R_1 \left(\frac{dR_1}{d\tau} \right)^2 + 4\eta_2 \frac{dR_1}{d\tau} + \frac{1}{\varepsilon \rho_2} \left\{ 2\sigma_2 + R_1 \left[p_2 - p'(T_2^{1\sigma}) \right] \right\} = 0 , \quad (32)$$

results in the famous integral-differential equation describing bubble dynamics. Furthermore, with several additional simplifications they come to an equation for the so called thermal controlled bubble growth

$$\Delta h \rho_1 \frac{dR_1}{d\tau} = \sqrt{\frac{3a_2}{\pi \tau}} \rho_2 c_{p2} [T_2 - T'(p)] . \quad (33)$$

Plesset and *Zwick* estimated also that the diffusion of non-condensable gases in the liquid is much slower process and therefore has not to be considered for intensive evaporation or condensation processes.

Dergarabedian (1953) introduces the surface tension of the momentum equation and considered the non-condensable gas component inside the vapour keeping its mass constant during the process and behaving isothermal

$$p_{1,n} = p_{10,n} (R_{10}/R_1)^3 .$$

He reported valuable bubble growth experiments for water used later by many authors. He used *Plesset* and *Zwick*'s solution for the surface temperature and then used a quadratic dependence between saturation temperature and saturation pressure. Then, the pressure inside the bubble was set to the saturation pressure and the momentum equation is integrated.

Forster and *Zuber* [8] followed the same approach as *Plesset* and *Zwick* and *Dergarabedian* but their solution of the liquid heat conduction problem was different

$$T_2^{1\sigma}(\tau) - T_2 = T_{20}^{1\sigma} - T_2 - \frac{\rho_1}{\rho_2} \frac{\Delta h}{c_{p2} \sqrt{\pi a_2}} \int_0^\tau \frac{R_1(\tau^*) \left(\frac{dR_1}{d\tau} \right)_{\tau^*}}{R_1(\tau) (\tau - \tau^*)^{1/2}} d\tau^*$$

$$\times \left\{ \exp \left[-\frac{[R_1(\tau) - R_1(\tau^*)]^2}{4a_2(\tau - \tau^*)} \right] - \exp \left[-\frac{[R_1(\tau) + R_1(\tau^*)]^2}{4a_2(\tau - \tau^*)} \right] \right\} d\tau^* \approx T_{20}^{1\sigma} - T_2 - \frac{\rho_1}{\rho_2} \frac{\Delta h}{c_{p2} \sqrt{\pi a_2}} \int_0^\tau \frac{(dR_1/d\tau)_{\tau^*}}{(\tau - \tau^*)^{1/2}} d\tau^* . \quad (34)$$

The results compared very well with the experiments by *Dergarabedian* for water with subcooling between 1.4 and 4.5 K. Finally the authors obtained the following approximation for bubble growth

$$\Delta h \rho_1 \frac{dR_1}{d\tau} = \sqrt{\frac{\pi a_2}{4\tau}} \rho_2 c_{p2} [T_2 - T'(p)] . \quad (35)$$

Scriven [31] compared both equations 33 and 35 with the data of *Dergarabedian* and concluded that *Plesset* and *Zwick*'s equation is in better agreement. The idea how to couple analytically, not numerically, the heat conduction problem with the momentum equation attracted the attention of *Mikic*, *Rohsenow* and *Griffith* [20]. They simplified the equation 1 to

$$\frac{3}{2} \left(\frac{dR_1}{d\tau} \right)^2 + \frac{p_2 - p_1}{\varepsilon \rho_2} = 0 \quad (36)$$

The pressure difference can be replaced by temperature difference using the linearized *Clausius* and *Clapeyron* equation

$$\frac{p_1 - p_2}{T'(p_1) - T'(p_2)} \approx \frac{dp'}{dT} = \frac{1}{T'(p_2)} \frac{\Delta h}{v'' - v'} \quad (37)$$

resulting in

$$\frac{1}{A^2} \left(\frac{dR_1}{d\tau} \right)^2 = 1 - \frac{T_2 - T'(p_1)}{T_2 - T'(p_2)}$$

where $A^2 = \frac{2}{3\varepsilon \rho_2} [T_2 - T'(p_2)] \frac{dp'}{dT}$. The *Plesset* and *Zwick* [24] equation (33) is then used. By setting $T_2^{1\sigma} = T'(p_1)$

$$\Delta h \rho_1 \frac{dR_1}{d\tau} = \sqrt{\frac{3a_2}{\pi \tau}} \rho_2 c_{p2} [T_2 - T'(p_1)] .$$

Excluding the interface temperature results in analytical expression for the time derivative of the bubble radius

$$\frac{1}{A^2} \left(\frac{dR_1}{d\tau} \right)^2 + \sqrt{\frac{\pi}{3a_2}} \frac{\Delta h \rho_1}{[T_2 - T'(p_2)] \rho_2 c_{p2}} \sqrt{\tau} \frac{dR_1}{d\tau} - 1 = 0 . \quad (38)$$

This result reduces to *Rayleigh*'s results with strong non-equilibrium and to the thermal controlled bubble size change for small non-equilibriums. In any case, the non-linearities like

non-monotonic temperatures profiles around the surface, surface tension, viscosity and second order terms in the momentum equation describing oscillation processes are excluded which is the main limitation of this, in many practical cases, very useful approach.

Coming back to the 60s: *Scriven* [31] formulated the problem already in 1959 with equations 1 and 2. In addition to the heat transfer he considered also the heat diffusion by the mass conservation equation having the same form as the *Fourier* equation. As already stated by *Plesset* and *Zwick*, the mass diffusion is much slower than heat diffusion. *Scriven* uses his own substitutions and come also to integral solution of the *Fourier* equation which was then estimated numerically for variety of situations. Later on this solution is serving as benchmark for some authors for checking their approximations. In his numerical estimate the coupling with equation 1 is not made. Therefore, dynamic oscillation effects can not be derived only from the heat transfer solution. I have to note that computationally the work for estimation of the *Scriven*'s integral is larger then for the solution of Eq. 19 and 20.

Zuber [39] discussed first the *Fritz* and *Ende* [9] solution

$$\Delta h \rho_1 \frac{dR_1}{d\tau} = \sqrt{\frac{a_2}{\pi \tau}} \rho_2 c_{p2} [T_2 - T'(p_1)] \quad (39)$$

and then considered growth and subsequent collapse in subcooled liquid near heated wall. With the observation that the thermal boundary layer was displaced by the growing bubble the authors come to the conclusion that as the bubble size reaches a maximum the condensation heat per unit time and unit wall surface almost equals the wall heat flux.

$$\sqrt{\pi a_2 \Delta \tau_{\max}} = \lambda_2 (T_w - T') / \dot{q}_w'' . \quad (40)$$

With this idea they obtained a maximum of the bubble size. The maximum radius is then computed by the thermal bubble growth formula. For the collapse phase the authors proposed to use Eq. 1 together with bubble growth equation in which the heat flux is reduced by the wall heat flux considered as a condensation heat flux. This mechanism is difficult to imagine because bubble growth and departure happens in much more complex circumstances and the collapse happens not at the place where the bubble was born.

Hewit and *Parker* [10] performed liquid nitrogen experiments and concluded that thermally controlled bubble growth models agree with data within a 25% error band but thermally controlled bubble collapse models do not represent satisfactorily the reality.

A complete numerical treatment was first made by *Theofanous* et al. [32, 33]. Equation 1 is used coupled with the thermal solution. The authors assumed a kinetic molecular transfer across the interface and come to the conclusion that the temperature is always close to the saturation temperature at the vapour pressure which was an important finding used later by many other authors including me. The solution of the *Fourier* equation was performed by assuming a thin boundary layer and monotonic quadratic temperature profile at the boundary.

$$T - T_2^{l\sigma} = (T_2 - T_2^{l\sigma}) \left[\frac{2(r - R_1)}{\delta} - \left(\frac{r - R_1}{\delta} \right)^2 \right], \quad (41)$$

with

$$T(R_1, \tau) = T_2^{1\sigma}, \quad T(R_1 + \delta, \tau) = T_2, \quad \left(\frac{\partial T}{\partial r} \right)_{R_1 + \delta} = 0, \quad (42)$$

as proposed before by *Bornhorst* and *Hatsopoulos* [3]. Inserting the profile in the energy equation and integrating between R_1 and $R_1 + \delta$ results in an ordinary differential equation connecting the change of the boundary layer thickness with time with the change of the bubble radius and of the bulk temperature. The so obtained equation was then included in the model instead of the *Fourier* equation. The model very well predicted the condensation and evaporation data obtained with superheats or subcoolings of few degrees which were the first proof that this direction of development is very fruitful. Such method was used also by *Casadei* [5] who used $(T - T_2^{1\sigma}) / (T_2 - T_2^{1\sigma}) = [(R_1 - r) / \delta]^2$ instead of Eq. 41. The limitations are in the assumptions made by resolving the temperature profile. *Lee* and *Merte* [19] used equations 1 and 2 exactly in the same form. They designed a *Runge-Kutta* method for solving the momentum equation and the *Patankar* finite volume method for resolving the temperature field dropping the assumption of monotonic profile. They also designed a non-equidistant grid having better resolution close to the interface. *Naude* and *Mendez* [41] reported that second order discretization in time and third order in space is needed to resolve the liquid temperature profile especially in the final collapse stage. The energy conservation equations for the bubble was not used by these authors and replaced with the saturation condition at the surface temperature at any time. This assumption does not allow having a non-equilibrium temperature in the bubble for very strong oscillations. This makes the main difference to our method. Therefore, the method I use here omits the disadvantages of assuming a monotonic profile in the liquid and not using the energy conservation for the vapour. This together with the simplicity of the numerical method is the main step forwards with respect to these authors.

In their detailed review of the state of the art up to 1977 *Plesset* and *Prosperetti* [26] used as a starting point the combined equation 1 and 2 as derived by *Plesset* and *Zwick* [24] to study oscillation behavior of such systems. Assuming that the gas state change is isentropic $p_{1,n} = p_{10,n} (R_{10} / R_1)^{3\kappa}$ the authors derived the Eigen frequency of oscillation to be

$$\omega_0^2 = \frac{3\kappa p}{\rho_2 R_{10}^2} - \frac{2\sigma}{\rho R_{10}^3}. \quad (43)$$

Jones and *Zuber* [12] pointed out that the vapour density change in Eq. 29 is important especially in pressure transients and proposed an approximation to compute the saturation vapour density following the surface temperature change. Further they used the *Carlsow* and *Jaeger* [4] solution of the *Fourier* equation for heat slab with variable temperature difference at the one boundary

$$T(x, \tau) - T_2 = \frac{x}{2\sqrt{\pi a_2}} \int_0^\tau (T_2^{1\sigma} - T_2) \frac{e^{-\frac{x^2}{4a_2(\tau-\eta)}}}{(\tau-\eta)^{3/2}} d\eta \quad (44)$$

to compute the heat flux at the bubble surface

$$\dot{q}_2^{n1\sigma} = k_s \frac{\lambda_2}{\sqrt{\pi a_2}} \left[\frac{\Delta T(0)}{\sqrt{\tau}} + \int_0^{\tau} \frac{d\tau}{\sqrt{\tau-\eta}} d\eta \right] \quad (45)$$

with $T_2 = \text{const}$, $\Delta T(\tau) = T_2^{1\sigma}(\tau) - T_2$, $\Delta T(0) = T'(p_{10}) - T_2$ and k_s the sphericity correction: $\pi/2$ by *Foster and Zuber*, $\sqrt{3}$ by *Plesset and Zwick*. Assuming different functions for the pressure change and therefore for the surface temperature change the authors integrated Eq. 45 analytically and derived expressions for the bubble size change without connection to the momentum equation. Interestingly, the authors obtained good comparisons for bubble growth in variable pressure field. A very similar approach was used by *Toda and Kitamura* [34]. They also allow for the vapour density to change and derive an integral solution for the thin boundary layer. *Alamgir* [2] proposed an analytical approximation for the density change combined with the *Jones and Zuber* solution. Again not coupling with the momentum equation they obtained acceptable accuracy for 4 experiments performed by them self with water at about 3 bar and superheating of about 4 to 6 K. *Wang and Bankoff* [36, 37] extended the *Jones and Zuber* [12] solution to bubble growth at the wall by modifying the equivalent volume of the bubble attached at the wall. Good comparison with their data for decompression experiments is reported. *Tsung-Chang and Bankoff* [35] considered the problem with linear pressure change and provided an approximate analytical solution to the *Plesset and Zwick* integral considering variable vapour density

$$T_2^{1\sigma}(\tau) - T_2 = \frac{\rho_1 \Delta h}{3\lambda_2} \sqrt{\frac{a_2}{\pi}} \int_0^{\tau} \frac{d\tau^* \left[\rho_1(\tau^*) R_1^3(\tau^*) \right]}{\left[\int_{\tau^*}^{\tau} R_1^4(\tau^{**}) d\tau^{**} \right]^{1/2}} d\tau^* . \quad (46)$$

Winters and Merte [38] are also interested in describing the bubble behaviour in variable pressure fields. The authors concluded that for bubble growth the monotonic temperature profile is a good approximation but for bubble collapse the temperature field should be considered as non monotonic. In this case they write the *Carlson and Jaeger* [4] approximate solution of the *Fourier* equation in the following form

$$\dot{q}_2^{n1\sigma} = k_s \frac{\lambda_2}{\sqrt{3a_2}} \left[\frac{T'(p_{10}) - T_2}{\sqrt{\tau}} + \sum_{m=1}^n \frac{T_2^{1\sigma}(\tau_m) - T_2^{1\sigma}(\tau_{m-1})}{\sqrt{\tau - \tau_m}} \right] . \quad (47)$$

Reynolds and Berthoud [29] solved the *Fourier* equation by making the same transformation as *Plesset and Zwick*

$$h = \frac{1}{3} (r^3 - R_1^3), \quad \bar{\tau} = \int_0^{\tau} [R_1(\tau^*)]^4 d\tau^*$$

and assuming a thin thermal boundary layer. Their result for the accumulated heat in J transferred from the surface into the liquid is

$$Q(R_1, \tau) = 4\lambda_2 \sqrt{\frac{\pi}{a_2}} \int_0^\tau \frac{R_1^4(\tau^*) [T_2^{1\sigma}(\tau) - T_2]}{\left[\int_{\tau^*}^\tau R_1^4(\tau^{**}) d\tau^{**} \right]^{1/2}} d\tau^* . \quad (48)$$

6. CONCLUSIONS

This work demonstrates how important is considering a bubble pressure separately from the system pressure as a dependent variable in multiphase fluid dynamics. Inserting this concept in future system computer codes will allow resolution of acoustical processes which are still not resolved in existing system computer codes widely used in the industry. The key problem in this step is fast and accurate solver of the transient Fourier equation around the bubble with variable boundary conditions.

APPENDIX 1: THE RAILEIGH-PLESSET EQUATION

Consider the liquid mass and momentum conservation equations valid in $R_1 < r < R_{2,cell}$, where

$$R_{2,cell} = [3/(4\pi n_1)]^{1/3} :$$

$$\text{Mass: } \frac{\partial \rho_2}{\partial \tau} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_2 u_2) = 0 , \quad (1-1)$$

$$\text{Momentum: } \frac{\partial u_2}{\partial \tau} + u_2 \frac{\partial u_2}{\partial r} + \frac{1}{\rho_2} \frac{\partial p_2}{\partial r} + 2\eta_2 \frac{\partial^2 u_2}{\partial r^2} = 0 . \quad (1-2)$$

The main idea by the derivation of the equation describing the bubble interface motion with respect to the centre, is to integrate the momentum equation along $R_1 < r < R_{2,cell}$. It will be assumed that at $R_{2,cell}$ the pressure is the liquid pressure. First I will present the known formalism for incompressible liquid and then I will extend the formalism for large time changes of pressure with the time.

Incompressible liquid: For incompressible liquid the mass conservation simplifies to $\partial(r^2 u_2)/\partial r = 0$ or

$$\frac{\partial r}{\partial \tau} = u_2(r) = \frac{R_1^2}{r^2} \frac{\partial R_1}{\partial \tau} . \quad (1-3)$$

Here the radius is considered to be only a function of time, $R_1 = R_1(\tau)$. *Rayleigh* (1917) was the first to equalized the expansion energy of a bubble from radius R_{10} to R_1

$$(p_1 - p_2) 4\pi \int_{R_{10}}^{R_1} r^2 dr = (p_1 - p_2) \frac{4}{3} \pi (R_1^3 - R_{10}^3) \quad (1-4)$$

to the energy consumed to change the kinetic energy of the surrounding liquid

$$\frac{1}{2} \rho_2 4\pi \int_{R_1}^{R_{cell}} u_2^2 r^2 dr = \frac{1}{2} \rho_2 4\pi R_1^4 \left(\frac{\partial R_1}{\partial \tau} \right)^2 \int_{R_1}^{R_{cell}} \frac{1}{r^2} dr = \frac{1}{2} \rho_2 4\pi R_1^4 \left(\frac{\partial R_1}{\partial \tau} \right)^2 \left(\frac{1}{R_1} - \frac{1}{R_{cell}} \right) \quad (1-5)$$

and obtained the equation used later in thousands of papers

$$\frac{3}{2} \left(\frac{\partial R_1}{\partial \tau} \right)^2 \left(1 - \frac{R_1}{R_{cell}} \right) + \frac{p_2 - p_1}{\rho_2} \left(1 - \frac{R_1^3}{R_{cell}^3} \right) = 0, \quad (1-6)$$

$$R_{cell} \rightarrow \infty.$$

Equation 1-6 is valid for no mass transfer at the surface. For mass transfer at the surface the interface velocity of the liquid is

$$u_2^{1\sigma} = u_1^{2\sigma} - \Delta u_{12}^\sigma = \left(1 - \frac{\rho_1^{2\sigma}}{\rho_2^{1\sigma}} \right) u_1^{2\sigma} = \left(1 - \frac{\rho_1^{2\sigma}}{\rho_2^{1\sigma}} \right) \frac{dR_1}{d\tau}. \quad (1-7)$$

Therefore the velocity at any radius in the liquid is related to the bubble radius derivative as follows

$$u_2 = \left(1 - \frac{\rho_1^{2\sigma}}{\rho_2^{1\sigma}} \right) \frac{R_1^2}{r^2} \frac{dR_1}{d\tau} = \varepsilon \frac{R_1^2}{r^2} \frac{dR_1}{d\tau}. \quad (1-8)$$

In equation 1-6 the history of the expansion that has to satisfy the momentum equation is not reflected. This history can be considered if the momentum equation in the liquid is integrated from R_1 to R_{cell} . For this purpose we need the participating liquid velocity first and second derivatives.

Assuming non compressible liquid they are obtained by differentiating Eq. (1-8) as follows:

$$\frac{\partial u_2}{\partial \tau} = \varepsilon \frac{R_1^2}{r^2} \frac{\partial^2 R_1}{\partial \tau^2} + 2\varepsilon R_1 \frac{1}{r^2} \left(\frac{\partial R_1}{\partial \tau} \right)^2 \quad (1-9)$$

$$\frac{\partial u_2}{\partial r} = -2\varepsilon \frac{\partial R_1}{\partial \tau} R_1^2 \frac{1}{r^3}, \quad (1-10)$$

$$\frac{\partial^2 u_2}{\partial r^2} = 6\varepsilon \frac{\partial R_1}{\partial \tau} R_1^2 \frac{1}{r^4}, \quad (1-11)$$

$$u_2 \frac{\partial u_2}{\partial r} = -2\varepsilon^2 R_1^4 \left(\frac{\partial R_1}{\partial \tau} \right)^2 \frac{1}{r^5}. \quad (1-12)$$

Replacing the velocity and its derivatives in the momentum equation 1-2 results in

$$\frac{R_1^2}{r^2} \frac{\partial^2 R_1}{\partial \tau^2} + 2 \left(R_1 \frac{1}{r^2} - \varepsilon R_1^4 \frac{1}{r^5} \right) \left(\frac{\partial R_1}{\partial \tau} \right)^2 + \frac{1}{\rho_2 \varepsilon} \frac{\partial p}{\partial r} + 12\eta_2 \frac{\partial R_1}{\partial \tau} R_1^2 \frac{1}{r^4} = 0. \quad (1-13)$$

Integrating between R_1 to $R_{2,cell}$

$$R_1^2 \frac{\partial^2 R_1}{\partial \tau^2} \int_{R_1}^{R_{2,cell}} \frac{1}{r^2} dr + 2 \left(R_1 \int_{R_1}^{R_{2,cell}} \frac{1}{r^2} dr - R_1^4 \int_{R_1}^{R_{2,cell}} \frac{1}{r^5} dr \right) \left(\frac{dR_1}{d\tau} \right)^2 + \frac{1}{\rho_2 \varepsilon} \int_{R_1}^{R_{2,cell}} dp_2 + 12\eta_2 \frac{\partial R_1}{\partial \tau} R_1^2 \int_{R_1}^{R_{2,cell}} \frac{1}{r^4} dr = 0 \quad (1-14)$$

results in

$$R_1 \left(1 - \frac{R_1}{R_{2,cell}} \right) \frac{\partial^2 R_1}{\partial \tau^2} + 2 \left(\frac{3}{4} - \frac{R_1}{R_{2,cell}} + \frac{1}{4} \frac{R_1^4}{R_{2,cell}^4} \right) \left(\frac{dR_1}{d\tau} \right)^2 + \frac{p_2 - p_1}{\rho_2 \varepsilon} + \frac{4\eta_2}{R_1} \frac{\partial R_1}{\partial \tau} \left(1 - \frac{R_1^3}{R_{2,cell}^3} \right) = 0. \quad (1-15)$$

For

$$R_1/R_{2,cell} = \alpha_1^{1/3}$$

this equation receives the form

$$R_1 (1 - \alpha_1^{1/3}) \frac{\partial^2 R_1}{\partial \tau^2} + 2 \left(\frac{3}{4} - \alpha_1^{1/3} + \frac{1}{4} \alpha_1^{4/3} \right) \left(\frac{dR_1}{d\tau} \right)^2 + \frac{p_2 - p_1}{\rho_2 \varepsilon} + (1 - \alpha_1) \frac{4\eta_2}{R_1} \frac{\partial R_1}{\partial \tau} = 0$$

used usually in the so called two pressure models of bubbly flow for analysing of acoustic processes e.g. *Akhmatov et al.* [1]. For $\alpha_1 \rightarrow 0$ which means $R_{2,cell} \rightarrow \infty$ the above equation reduces to

$$R_1 \frac{\partial^2 R_1}{\partial \tau^2} + \frac{3}{2} \left(\frac{\partial R_1}{\partial \tau} \right)^2 + \frac{p_2 - p_1}{\rho_2 \varepsilon} + \frac{4\eta_2}{R_1} \frac{\partial R_1}{\partial \tau} = 0 \quad (1-16)$$

which is the so called *Rayleigh-Plesset* equation used by many authors, compare with Eq. 10 in *Scriven* [31].

Hijikata et al. ([11] used in a one dimensional model the *Rayleigh-Plesset* equation by replacing $dR_1/d\tau = u_1 dR_1/dx$. Following this idea

$$\frac{dR_1}{d\tau} = \frac{\partial R_1}{\partial \tau} + u_1 \nabla R_1 \quad (1-17)$$

and therefore

$$R_1 \left(\frac{\partial^2 R_1}{\partial \tau^2} + u_1 \frac{\partial}{\partial \tau} \nabla R_1 + \nabla R_1 \frac{\partial u_1}{\partial \tau} \right) + \frac{3}{2} \left(\frac{\partial R_1}{\partial \tau} + u_1 \nabla R_1 \right)^2 + \frac{p_2 - p_1}{\rho_2 \varepsilon} + \frac{4\eta_2}{R_1} \frac{\partial R_1}{\partial \tau} = 0. \quad (1-18)$$

Prosperetti and Crum [27], *Kamath and Prosperetti* [13] tuck into account some compressibility effect in the liquid by deriving the equation describing the bubble radius dynamics.

Compressible liquid: The complete system 1-1 and 1-2 together with the energy conservation in the liquid has to be solved in this case. Replacing the density differentials in the mass

conservation equation with their equals from the differential form of the equation of state results in

$$\frac{1}{\rho_2 a_2^2} \left(\frac{\partial p_2}{\partial \tau} + u_2 \frac{\partial p_2}{\partial r} \right) + \frac{1}{\rho_2} \frac{\partial \rho_2}{\partial s_2} \left(\frac{\partial s_2}{\partial \tau} + u_2 \frac{\partial s_2}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_2) = 0.$$

Here, only the case for large pressure changes with the time will be considered resulting in

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_2) = - \frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau}.$$

As in the case of incompressible liquid we have to derive from these equations the velocity and its derivatives, to replace in the momentum equation and to integrate it from the interface to the end of the cell. The velocity and the derivative are

$$u_2 = \varepsilon \frac{R_1^2}{r^2} \frac{dR_1}{d\tau} - \frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau} \frac{1}{3} (r^3 - R_1^3)$$

$$\frac{\partial u_2}{\partial r} = - \frac{2}{r} u_2 - \frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau} = - 2\varepsilon \frac{R_1^2}{r^3} \frac{dR_1}{d\tau} - \frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau}$$

$$\frac{\partial^2 u_2}{\partial r^2} = 6\varepsilon \frac{R_1^2}{r^4} \frac{dR_1}{d\tau}$$

$$\frac{\partial u_2}{\partial \tau} = \varepsilon \frac{R_1^2}{r^2} \frac{\partial^2 R_1}{\partial \tau^2} + 2\varepsilon R_1 \frac{1}{r^2} \left(\frac{\partial R_1}{\partial \tau} \right)^2 - \frac{1}{\rho_2 a_2^2} \frac{\partial^2 p_2}{\partial \tau^2} \frac{1}{3} (r^3 - R_1^3)$$

and therefore

$$u_2 \frac{\partial u_2}{\partial r} = - 2\varepsilon^2 \frac{R_1^4}{r^5} \left(\frac{dR_1}{d\tau} \right)^2 - \varepsilon \frac{R_1^2}{r^2} \frac{dR_1}{d\tau} \frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau} + 2\varepsilon \frac{1}{3} (r^3 - R_1^3) \frac{R_1^2}{r^3} \frac{dR_1}{d\tau} \frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau} + \left(\frac{1}{\rho_2 a_2^2} \frac{\partial p_2}{\partial \tau} \right)^2 \frac{1}{3} (r^3 - R_1^3).$$

Replacing in the momentum equation I obtain

$$\frac{R_1^2}{r^2} \frac{\partial^2 R_1}{\partial \tau^2} + 2 \left(R_1 \frac{1}{r^2} - \varepsilon R_1^4 \frac{1}{r^5} \right) \left(\frac{\partial R_1}{\partial \tau} \right)^2 + \frac{1}{\rho_2 \varepsilon} \frac{\partial p}{\partial r} + 12\eta_2 \frac{\partial^2 u_2}{\partial r^2} + \left\{ \left(\frac{2}{3} - \frac{1}{r^2} - \frac{2}{3} \frac{R_1^3}{r^3} \right) R_1^2 \frac{dR_1}{d\tau} \frac{\partial p_2}{\partial \tau} \right. \\ \left. + \frac{1}{3\varepsilon} (r^3 - R_1^3) \left[\frac{1}{\rho_2 a_2^2} \left(\frac{\partial p_2}{\partial \tau} \right)^2 - \frac{\partial^2 p_2}{\partial \tau^2} \right] \right\} \frac{1}{\rho_2 a_2^2} = 0$$

After integration from the surface of the bubble to the end of the cell the momentum equation receives the form:

$$R_1 \left(1 - \frac{R_1}{R_{cell}} \right) \frac{\partial^2 R_1}{\partial \tau^2} + 2 \left(\frac{3}{4} - \frac{R_1}{R_{cell}} + \frac{1}{4} \frac{R_1^4}{R_{cell}^4} \right) \left(\frac{dR_1}{d\tau} \right)^2 + \frac{p_2 - p_1}{\rho_2 \varepsilon} + \frac{4\eta_2}{R_1} \frac{\partial R_1}{\partial \tau} \left(1 - \frac{R_1^3}{R_{cell}^3} \right) \\ \left\{ \left(\frac{2}{3} R_{cell} \left(1 - \frac{R_1}{R_{cell}} \right) - \frac{1}{R_1} \left(1 - \frac{R_1}{R_{cell}} \right) - \frac{1}{3} R_1 \left(1 - \frac{R_1^2}{R_{cell}^2} \right) \right) R_1^2 \frac{dR_1}{d\tau} \frac{\partial p_2}{\partial \tau} \right\} \frac{1}{\rho_2 a_2^2} = 0 \\ \left\{ + \frac{R_{cell}^4}{3\varepsilon} \left(\frac{1}{4} - \frac{R_1^3}{R_{cell}^3} + \frac{3}{4} \frac{R_1^4}{R_{cell}^4} \right) \left[\frac{1}{\rho_2 a_2^2} \left(\frac{\partial p_2}{\partial \tau} \right)^2 - \frac{\partial^2 p_2}{\partial \tau^2} \right] \right\}$$

Again using $R_1/R_{cell} = \alpha_1^{1/3}$ results in

$$R_1 \left(1 - \alpha_1^{1/3} \right) \frac{\partial^2 R_1}{\partial \tau^2} + 2 \left(\frac{3}{4} - \alpha_1^{1/3} + \frac{1}{4} \alpha_1^{4/3} \right) \left(\frac{dR_1}{d\tau} \right)^2 + \frac{p_2 - p_1}{\rho_2 \varepsilon} + (1 - \alpha_1) \frac{4\eta_2}{R_1} \frac{\partial R_1}{\partial \tau} + f \left(\frac{\partial p_2}{\partial \tau}, \frac{\partial^2 p_2}{\partial \tau^2} \right) = 0 \\ f \left(\frac{\partial p_2}{\partial \tau}, \frac{\partial^2 p_2}{\partial \tau^2} \right) = \frac{1}{\rho_2 a_2^2} \left\{ \left((1 - \alpha_1^{1/3}) \left(\frac{2}{3} R_{cell} - \frac{1}{R_1} \right) - \frac{1}{3} R_1 (1 - \alpha_1^{2/3}) \right) R_1^2 \frac{dR_1}{d\tau} \frac{\partial p_2}{\partial \tau} \right\} \\ \left\{ + \frac{R_{cell}^4}{3\varepsilon} \left(\frac{1}{4} - \alpha_1 + \frac{3}{4} \alpha_1^{4/3} \right) \left[\frac{1}{\rho_2 a_2^2} \left(\frac{\partial p_2}{\partial \tau} \right)^2 - \frac{\partial^2 p_2}{\partial \tau^2} \right] \right\}$$

APPENDIX 2: THE LIQUID CONSERVATION EQUATION

The energy conservation of incompressible liquid in a coordinate system with origin the center of the bubble without internal heat sources is

$$\frac{\partial T_2}{\partial \tau} + \dot{r} \frac{\partial T_2}{\partial r} - \frac{a_2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T_2}{\partial r} \right) = 0. \quad (2-1)$$

Here the liquid temperature is a function of time and radius $T_2 = T_2(\tau, r)$ and therefore

$$dT_2 = \frac{\partial T_2}{\partial \tau} d\tau + \frac{\partial T_2}{\partial r} dr. \quad (2-2)$$

Setting the coordinate system at the interface,

$$x = \frac{r - R(\tau)}{\delta}, \quad (2-3)$$

where $0 \leq x \leq 1$, $R_1(\tau) \leq r \leq R_1(\tau) + \delta$, $r = \delta x + R(\tau)$ the temperature is then a function of the time and the distance from the surface

$$T_2 = T_2[\tau, r = \delta x + R(\tau)] = T_2(\tau, x)$$

and

$$dT_2 = \left(\frac{\partial T_2}{\partial \tau} + \dot{r} \frac{\partial T_2}{\partial r} \right) d\tau + \delta \frac{\partial T_2}{\partial r} dx. \quad (2-4)$$

Comparing equations 2-2 and 2-4 we obtain

$$\left(\frac{\partial T_2}{\partial \tau} \right)_x = \frac{\partial T_2}{\partial \tau} + \dot{r} \frac{\partial T_2}{\partial r}, \quad \left(\frac{\partial T_2}{\partial x} \right)_r = \delta \frac{\partial T_2}{\partial r}. \quad (2-5, 6)$$

Solving with respect to the original derivatives we have

$$\frac{\partial T_2}{\partial r} = \frac{1}{\delta} \left(\frac{\partial T_2}{\partial x} \right)_r, \quad \frac{\partial T_2}{\partial \tau} = \left(\frac{\partial T_2}{\partial \tau} \right)_x - \dot{r} \left(\frac{\partial T_2}{\partial x} \right)_r, \quad \frac{\partial r}{\partial \tau} = x \frac{d\delta}{d\tau} + \frac{dR_1}{d\tau}. \quad (2-7-8, 9, 10)$$

Applying to the original energy conservation equation for $\delta = \text{const}$ results in

$$\left(\frac{\partial T_2}{\partial \tau} \right)_x - \frac{a_2}{r^2} \frac{1}{\delta} \left(\frac{\partial}{\partial x} \right)_r \left[r^2 \frac{1}{\delta} \left(\frac{\partial T_2}{\partial x} \right)_r \right] = 0, \quad (2-11)$$

or for $\delta = 1$

$$\left(\frac{\partial T_2}{\partial \tau} \right)_x - \frac{a_2}{r^2} \left(\frac{\partial}{\partial x} \right)_r \left[r^2 \left(\frac{\partial T_2}{\partial x} \right)_r \right] = 0, \quad (2-12)$$

see *Lee and Merte* [19]. As already mentioned we see that the convection term disappears.

APPENDIX 3: PRESSURE EQUATION FOR THE BUBBLE

Equation 3-1 is the so called pressure equation or volume conservation equation for the bubble. It is in fact Eq. 5.207 in *Kolev* [17] written for single gas sphere

$$\frac{1}{\rho_1 a_1^2} \frac{dp}{d\tau} = -\frac{1}{\rho_1} \frac{3}{R_1} (\rho w)_2^{1\sigma} - \frac{1}{T_1} \frac{1}{\rho_1^2} \left(\frac{\partial \rho_1}{\partial s_1} \right)_p \frac{3}{R_1} \left\{ \dot{q}_1^{r2\sigma} - (\rho w)_2^{1\sigma} c_{p1} [T'(p_1) - T_1] \right\} - \frac{1}{\gamma_v} \frac{d\gamma_v}{d\tau}. \quad (3-1)$$

Here

$$\left(\frac{\partial \rho_1}{\partial s_1} \right)_p = \left(\frac{\partial \rho_1}{\partial T_1} \right)_p \frac{T_1}{c_{p1}}. \quad (3-2)$$

$$\left(\frac{\partial \rho_1}{\partial T_1} \right)_p = -\frac{\rho_1}{T_1}. \quad (3-3)$$

γ_v is the ratio of the bubble volume to the initial volume

$$\gamma_v = \frac{V_1}{V_{10}} = \left(\frac{R_1}{R_{10}} \right)^3, \quad (3-4)$$

and therefore

$$\frac{1}{\gamma_v} \frac{d\gamma_v}{d\tau} = \frac{3}{R_1} \frac{dR_1}{d\tau}. \quad (3-5)$$

The change of the bubble radius with the time consists of the mechanical component dictated by the force balance $(dR_1/d\tau)_{mom}$ and of a component depending on the mass transfer as follows

$$\frac{dR_1}{d\tau} = \left(\frac{dR_1}{d\tau} \right)_{mom} - \frac{1}{\rho_2} (\rho w)_2^{1\sigma}. \quad (3-6)$$

Replacing 3-2, 3, 5, 6 into 3-1 results in

$$\frac{1}{\rho_1 a_1^2} \frac{dp}{d\tau} = -\frac{3}{R_1} \left[\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) (\rho w)_2^{1\sigma} + \left(\frac{dR_1}{d\tau} \right)_{mom} \right] + \frac{1}{\rho_1 c_{p1} T_1} \frac{3}{R_1} \left\{ \dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} [T'(p_1) - T_1] \right\}. \quad (3-7)$$

Neglecting the mass transfer and assuming perfect gas this equation reduces to the one derived by *Prosperetti* and *Crum* [27], *Kamath* and *Prosperetti* [13] from the mass conservation. The heat transfer contribution on the bubble pressure change

$$= \dots + \frac{3}{R_1} \frac{\kappa - 1}{\kappa} \frac{\dot{q}_1^{n2\sigma}}{p_1}$$

is taken in this form into account by many authors e.g. *Prosperetti* and *Crum* [27], *Kamath* and *Prosperetti* [13], *Nigmatulin* [23], *Kim* [14, 15, 16]. Defining the pseudo-isentropic exponent as $\kappa = \rho/(p a^2)$ and rearranging I obtain

$$d \ln \left(p_1^{\frac{1}{\kappa}} R_1^3 \right) = \frac{3}{R_1} \left[- \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) (\rho w)_2^{1\sigma} + \frac{\dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} [T'(p_1) - T_1]}{\rho_1 c_{p1} T_1} \right] d\tau \quad (3-8)$$

or in finite difference form

$$p_1 \approx p_{10} \left(\frac{R_{10}}{R_1} \right)^{3\kappa} \exp \left\{ \frac{3\kappa \Delta \tau}{R_1} \left[- \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) (\rho w)_2^{1\sigma} + \frac{\dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} [T'(p_1) - T_1]}{\rho_1 c_{p1} T_1} \right] \right\} \quad (3-9)$$

Compared with Eq. 3-7 this equation has the advantage that if numerically evaluated the pressure in the bubble remains always positive.

For no heat and mass transfer across the interface the equation reduces to the well known isentropic pressure volume relation $p_1 \approx p_{10} (R_{10}/R_1)^{3\kappa}$, *Plesset* and *Zwick* [24] or for isothermal state change $p_1 \approx p_{10} (R_{10}/R_1)^3$, *Dergarabedian* (1953). For isothermal state change of bubbles with constant number density $p_1 \approx p_{10} (\alpha_{10}/\alpha_1)^\kappa$.

APPENDIX 4: BUBBLE ENERGY CONSERVATION

The pseudo-entropy form of the energy conservation is

$$\rho_1 c_{p1} \frac{dT_1}{d\tau} - \frac{dp_1}{d\tau} = \frac{3}{R_1} \left[\dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} (T_1^{2\sigma} - T_1) \right]. \quad (4-1)$$

Defining the pseudo isentropic exponent $\kappa = \rho_1 a_1^2 / p_1 \approx c_{p1} / (c_{p1} - R_1)$ and rearranging I obtain

$$T_1 \approx T_{10} \left(\frac{p_1}{p_0} \right)^{(\kappa-1)/\kappa} \times \exp \left\{ \frac{\kappa-1}{\kappa} \frac{3\Delta\tau}{R_1 p_1} \left[\dot{q}_1^{n2\sigma} - (\rho w)_2^{1\sigma} c_{p1} (T'(p_1) - T_1) \right] \right\}. \quad (4-2)$$

Prosperetti and *Crum* [27], *Kamath* and *Prosperetti* [13] solved numerically the gas energy conservation equation for the case of no mass transfer at the interface

$$\rho_1 c_{p1} \left(\frac{\partial T_1}{\partial \tau} + u_1 \frac{\partial T_1}{\partial r} \right) - \frac{\partial p_1}{\partial \tau} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \lambda_1 \frac{\partial T_1}{\partial r} \right) = 0 \quad (4-3)$$

by considered an uniform pressure inside the bubble in order to compute accurately the bubble side heat transfer. Considering processes without mass transfer the liquid energy conservation equation was neglected in this case because the main heat transfer resistance is at the gas side. This scheme was combined with homogeneous one dimensional model of a bubbly flow by *Kim* [14, 15, 16]. *Kim* has demonstrated that important non-linear effects of the wave propagation can be considered.

7. NOMENCLATURE

Latin

a	temperature conductivity, m/s
c_p	specific capacity at constant pressure, J/(kgK)
D_1	bubble diameter, m
D_3	droplet diameter, m
htc	heat transfer coefficient, W/(m ² K)
h	specific enthalpy, J/kg
Nu	$= htc 2R_1 / \lambda$, Nusselt number, dimensionless
Pe	Peclet number, dimensionless
p	pressure
Q	accumulated heat transferred from the surface into the liquid, J
$\dot{q}_1^{n2\sigma}$	heat flux from the liquid interface into the gas, W/m ²

$\dot{q}_2^{n1\sigma}$	heat flux from the gas interface into the liquid, W/m ²
R_1	bubble radius, m
r	radius, m
T	temperature, K
u	radial velocity, m/s
x	$= r - R_1(\tau)$, distance from the bubble surface, m

Greek

β	$= (R_1/R_{10})^3$, dimensionless
ε	density ratio, dimensionless
γ_v	$(R_1/R_{10})^3$, ratio of the bubble volume to the initial volume, dimensionless
$\Delta\tau$	time step, s
Δh	evaporation specific enthalpy, J/kg
Δu_{12}^σ	$= u_1^{2\sigma} - u_2^{1\sigma}$, difference between the gas interface velocity and the liquid interface velocity.
ΔV_{12}	bubble velocity minus liquid velocity, m/s
δ	normalizing length, m
κ	isentropic exponent, dimensionless
λ	thermal conductivity, W/(mK)
η	dynamic viscosity, kg/(ms)
ω_0	eigen frequency of bubble oscillation, 1/s
σ	surface tension, N/m
ρ	density, kg/m ³
$(\rho w)_1^{2\sigma}$	mass flow rate from the liquid interface into the gas, kg/(m ² s)
$(\rho w)_2^{1\sigma}$	mass flow rate from the gas interface into the liquid, kg/(m ² s)
τ	time, s

Subscripts

1	Gas, bubble
2	Liquid
0	Initial
mom	computed by use of the momentum equation

Superscripts

'	saturated liquid
"	saturated vapour
$_{2}^{1\sigma}$	inside the liquid interface to the gas
$_{1}^{2\sigma}$	inside gas interface to the liquid

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