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Flow stability of liquid metal flow under transverse magnetic field

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Abstract

A stability analysis of a viscous incompressible liquid metal flow in an annular linear induction electromagnetic pump for sodium coolant circulation of LMR (Liquid Metal Reacters) is carried out when transverse magnetic fields permeate an electrically conducting sodium fluid across the narrow annular gap. Due to a negligible skin effect, the radial magnetic field is assumed to be constant over the narrow channel gap, and the steady state solution of an axial velocity is obtained as a function of radius r. Small perturbations for MHD fields in the form of $\mathbf{f}(r)e^{j(\omega t-\mathbf{k}\cdot\mathbf{r})}$, where ω is the angular frequency and $\mathbf{k}$ is the wave vector of perturbation, are considered and perturbed MHD equations are linearized. The solutions of the perturbed equations are sought in the form of linear combination of independent orthogonal functions $\{\psi_n(\zeta)_{n=0}^{\infty}\}$ in the non-dimensional radial interval (0,1) and each orthogonal function is chosen to satisfy boundary conditions of adhesion at the solid walls of the channel. Under assumption that solutions of the equations are not oscillated rapidly according to radial coordinate r, finite numbers of orthogonal polynomials are considered. As a result, simultaneous equations with coefficients of steady-state solutions are arranged and dispersion relations between angular frequency and wave number of perturbed state are sought. The imaginary part of the angular frequency (ω_i) is taken into consideration from the condition of the existence of nontrivial solution of the system, which yields the relation between critical Reynolds number (Re_{cr}) and Hartmann number (H_a). In the present study, critical Reynolds number and Wave numbers are plotted on the Hartmann number for long wave perturbation, thus, it is shown that a magnetic field has a significant stabilizing effect on liquid metal flow.

1 Introduction

Annular linear induction electromagnetic pumps[ALIP] have been widely investigated for the transportation of sodium-coolant in the [LMR][1−6]. The electrically conducting liquid metal, in an annular channel of ALIP, suffers from an axial electromagnetic force(Lorentz force) generated by induced azimuthal electric current and a radial magnetic field. Electromagnetic forces in the fluid give a dominant contribution to the force balance due to the strong applied magnetic field. For most electromagnetic pump systems with high electrical conductivity under strong magnetic fields, viscous forces are very weak compared with electromagnetic forces as anticipated by high Hartmann number(H_a). In fact, viscous effects are confined to thin boundary layers near the annular channel walls in laminar flows with adhesive boundary conditions for the velocity. In the electromagnetic-force-dominant MHD laminar flow system with a sufficiently high H_a, the radial profile of the velocity distribution turns flat so that velocity gradients become zero in the entire flow region with the exception of the narrow wall layers[7]. It is indicated that the flow with such a velocity distribution would be more stable than the flow with a parabolic velocity distribution[7]. In the present study, for a simple model of axially-infinite electromagnetic pump with equivalent current sheet, the steady-state solution of unidirectional velocity distribution on the laminar flow at very narrow annular channel gap is found under a transverse magnetic field produced by three-phase magnet coils. Using the method of small perturbation for the fluid velocity, magnetic field and pressure, linear stability analysis is carried out on the flow of an incompressible liquid metal flow. The criterion for stable operation is sought from the imaginary part of perturbed angular frequency and the critical Reynolds number(R_{ecr}) is thus found in terms of Hartmann number(H_a) for the pump with flowrate of 40 l/min.

2 Steady-state solution

The schematic of an analytical electromagnetic pump model with equivalent sheet current is depicted in Fig. 1 as an idealization of the MHD induction flow system with a practical three-phase coil arrangement. The annular channel consists of a narrow gap between two infinite coaxial cylinders with different radius. The liquid metal flow of high electrical conductivity is assumed to be laminar, incompressible and axisymmetric with axial velocity components depending on radial positions. The pumping fluid is characterized by its density ρ, viscosity μ, electrical conductivity σ, and vacuum magnetic permeability μ_0. Magnet cores outside the annular channel are the idealized silicon-iron laminations with zero conductivity and infinite permeability. Applied electrical current of magnet coils is given by sinusoidal equivalent sheet current that flows azimuthally on the outer wall(r_b), and it generates a radial magnetic field with an axial magnetic field passing through the inner core(r_a). This current is represented by the peak line current density J_m given by $\frac{3\sqrt{2}k_w NI}{p\tau}$ where k_w, N, I, p, and τ are the winding coefficient, turns of coil, input current, pole pairs, and pole pitch, respectively. With the help of angular frequency ω' and wave number k' of travelling current, the sheet current J_a is thus described as[8]

$$\mathbf{J}_a(r_b, z, t) = Re[J_m e^{j(\omega' t - k' z)}]\hat{\theta} \tag{1}$$

Magnetic and electric fields and electrical current induced by sinusoidal applied sheet current with axisymmetry are generated also in the form of sinusoidal and axisymmetric fields having different phases of each other. Mathematical descriptions for these fields are[9]

$$\mathbf{F} = Re\left[\left(F_r(r)\hat{\mathbf{r}} + F_z(r)\hat{\mathbf{z}} \right) e^{j(\omega' t - k' z)} \right] \tag{2}$$

where $\mathbf{F} = \mathbf{B}$, $\mathbf{E}$, and $\mathbf{J}$. Now, the conducting incompressible fluid under time-varing magnetic field is governed by the following set of dimensionless MHD equations:

$$\nabla \cdot \mathbf{V} = 0 \tag{3}$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla P + \frac{1}{R_e}\nabla^2\mathbf{V} + \frac{{H_a}^2}{R_e}\mathbf{J}\times\mathbf{B} \tag{4}$$

$$\nabla \times \mathbf{B} = R_m \mathbf{J} \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{7}$$

$$\mathbf{J} = (\mathbf{E} + \mathbf{V}\times\mathbf{B}) \tag{8}$$

where $\mathbf{V}$, P, $\mathbf{J}$, $\mathbf{B}$, and $\mathbf{E}$ denote dimensionless quantities of velocity, pressure, current density, magnetic and electric fields, normalized by their reference values with U_0, ρU_0^2, $\sigma U_0 B_0$, B_0, and $U_0 B_0$, respectively. Here, $B_0 = \frac{\mu_0 J_m}{\sqrt{2}R_0 k'}$ and $U_0 = \frac{\omega'}{k'}(1-s)$ when the flow speed is characterized by the slip s with respect to the synchronous speed $\frac{\omega'}{k'}$. Then, the equations are characterized by Reynolds number, ($R_e = \frac{\rho R_0 V_0}{\mu}$), Hartmann number ($H_a = \sqrt{\frac{\sigma B_0^2 R_0^2}{\mu}}$), magnetic Reynolds number($R_m = \mu_0 \sigma R_0 V_0$). In this system, it is assumed that the induced magnetic field is negligible due to small R_m, and the radial magnetic field is constant over narrow channel gap by negligible skin effect, and the pressure gradient is constant. Using Eqs. (2) through (8), the axial component of the force balance for the time averaged quantities yields[9]

$$-\frac{\partial P}{\partial z} + \frac{1}{R_e}\frac{\partial}{r\partial r}\left(r\frac{\partial u}{\partial r} \right) + \frac{\beta^2 H_a^2}{2R_e}(1-u) = 0 \tag{9}$$

where

$$\beta^2 = \frac{1}{1 + \left(\frac{R_m s}{k' R_o} \right)^2} \tag{10}$$

Here, $s \equiv \int_{r_a}^{r_b}(1-u)dr$ means the average slip over annular channel. General solutions of Eq. (9) are simply linear combinations of modified Bessel functions of the 1st and 2nd kinds. Applying boundary conditions at the walls for the velocity given by

$$u(r_a) = u(r_b) = 0 \tag{11}$$

to the general solution, the exact solution for the axial velocity is found as[11]

$$u = \gamma\left(1 - \frac{1}{I_o(\alpha r_a)K_o(\alpha r_b) - K_o(\alpha r_a)I_o(\alpha r_b)}\right) \\ \times\left[(K_o(\alpha r_b) - K_o(\alpha r_a))I_o(\alpha r) - (I_o(\alpha r_a) - I_o(\alpha r_b))K_o(\alpha r)\right] \qquad (12)$$

where

$$\gamma = -\frac{R_e}{\alpha^2}\frac{\partial P}{\partial z} + 1 = -\frac{R_e}{\alpha^2}\frac{\eta(1-s)^2}{4} + 1 \\ \alpha = \beta H_a, \qquad \eta:\ \text{friction coefficient} \qquad (13)$$

3 Stability of liquid metal flow in the narrow channel

Sodium fluid of ALIP is developed in the axial direction under a traveling moving magnetic field with an induced current flow. Then, by some perturbations, the system may experience turbulence[7] and, eventually, flow separation or local cavitation. In addition, abrupt increase of induced current can lead to damage of the system by growing perturbation. As generally known, a magnetic field improves hydrodynamic stability of the flow or suppresses a turbulence already present[8]. ALIP mainly suffers from axial developing force by the Lorentz's product of azimuthal induced current and radial magnetic field, while experiences radial force by axial magnetic field which exists in the liquid metal. Generally, perturbation can be caused in the every direction, but in the present study, the stability effect of the magnetic field on the liquid metal flow is estimated through the analysis of the axisymmetric two dimensional linear stability taking into account the perturbation on the developing direction. To analyze a magnetic field effect on an electrically conducting flow, critical Reynolds number $R_{e_c}r$ is plotted according to Hartmann number H_a solving 4th Orr-Sommerfeld equation[11–15] in case of small magnetic Reynold number, R_m. Firstly, because $\alpha r_b > \alpha r > \alpha r_a \gg 1$ in the steady state solution given by Eq. (12), mathematical convenience with form of hyperbolic type[16] is obtained by the help of approximation of Bessel function. Each dimensionless perturbed physical quantities $\mathbf{V}$, $\mathbf{B}$ and P can be written in the Fourier sum of an initial value and the superposed perturbations of all normal modes as follows[17]:

$$\mathbf{V}(\mathbf{r},t) = \mathbf{V}_0 + \sum_{k,m}\mathbf{v}(\mathbf{r})e^{j(\omega t - kz - m\theta)} \\ \mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 + \sum_{k,m}\mathbf{b}(\mathbf{r})e^{j(\omega t - kz - m\theta)} \qquad (14) \\ P(\mathbf{r},t) = P_0 + \sum_{k,m}p(r)e^{j(\omega t - kz - m\theta)}$$

where $\mathbf{V_0}$, $\mathbf{B_0}$ and P_0 are steady state values, ω perturbed angular frequency, k perturbed axial directional wave number and m perturbed azimuthal wave number. Substituting Eq. (14) into Eqs. (4) - (8) yields following linear perturbation equations after neglecting quadratic terms:

$$\begin{aligned}&\frac{\partial \mathbf{v}(r)}{\partial t} + (\mathbf{V_0}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{V_0}\\ &= \nabla\prod + \frac{1}{R_e}\nabla^2\mathbf{v} + \frac{{H_a}^2}{R_e R_m}(\mathbf{B_0}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{B_0})\end{aligned} \tag{15}$$

$$\frac{\partial \mathbf{b}(r)}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B_0} - (\mathbf{B_0}\cdot\nabla)\mathbf{v} = \frac{1}{R_m}\nabla^2\mathbf{b} \tag{16}$$

where

$$V_0 = u(r)\hat{\mathbf{z}}$$

$$\prod = p + \frac{H_a^2}{R_e R_m}\quad(\mathbf{B_0}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{B_0})$$

For negligible $R_m << 1$, magnetic field is expanded in series of R_m[11]

$$\mathbf{B} = \mathbf{B_0} + R_m\mathbf{b}(\mathbf{r}, t) + O(R_m^2).$$

Leaving zero-order terms on R_m in Eqs.(15) and (16) and operating on Eq.(15) by $\hat{\mathbf{r}}\cdot\nabla\times\nabla\times$ with further assumption of $B_r \gg B_z$ for sufficiently large H_a in narrow gap and using Eq. (16), pressure and magnetic field terms can be suitably removed and then, we have representation on velocity as follows:

$$\begin{aligned}&\frac{\partial}{\partial t}\nabla^2 v_r + \hat{\mathbf{r}}\cdot(\nabla^2 - \nabla\nabla\cdot)(V_0\frac{\partial \mathbf{v}}{\partial \mathbf{z}} + v_r\frac{\partial \mathbf{V_0}}{\partial \mathbf{r}} + v_z\frac{\partial \mathbf{V_0}}{\partial \mathbf{z}})\\ &+\frac{H_a^2}{R_e}\frac{\partial^2 v_r}{\partial r^2} - \frac{1}{R_e}\nabla^4 v_r = 0\end{aligned} \tag{17}$$

Introducing stream function for the perturbed velocity given by

$$\Psi = \sum_{k,m}\psi(r)e^{j(\omega t - kz - m\theta)} \tag{18}$$

we have perturbed velocity from the curl operation on the stream function as follows:

$$\mathbf{v} = (-\frac{\partial\Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial}{\partial r}r\Psi) \tag{19}$$

Then, for axisymmetric ($m = 0$) perturbed MHD flow, substituting Eqs. (18) and (19) into Eq. (17) gives Orr-Sommerfeld equation on the r as follows:

$$\frac{\partial^4\Psi}{\partial r^4} + \bar{A}\frac{\partial^3\Psi}{\partial r^3} + \bar{B}\frac{\partial^2\Psi}{\partial r^2} + \bar{C}\frac{\partial\Psi}{\partial r} + \bar{D}\Psi = 0 \tag{20}$$

where

$$\begin{aligned}
\bar{A} &= \frac{2}{r},\\
\bar{B} &= 2k^2{R_0}^2 - R_e(V_0 k\, R_0 - \omega\frac{R_0}{V_0})j - H_a^2 - \frac{1}{r^2}\\
\bar{C} &= \frac{1}{r^3} + 2k^2{R_0}^2 - R_e(V_0\, k^3\, {R_0}^3 j - k^2 R_0 \omega\frac{R_0}{V_o} + V_0 k r_0 j)\\
\bar{D} &= k^4{R_0}^4 + (V_0 k R_0 - \omega\, \frac{R_0}{V_0})k^2{R_0}^2 j + V_0'' R_e j k R_0
\end{aligned} \tag{21}$$

Eq.(20) is solved with no-slip boundary conditions with the velocity at the walls given to be

$$\psi(r_a) = \psi(r_b) = 0, \qquad \psi'(r_a) = \psi'(r_b) = 0 \tag{22}$$

$\psi(r)$ can be represented from Forurier expansion as follows:

$$\psi(r) = a_0 + \sum_{n=0}^{\infty} a_n \sin n\pi r + \sum_{n=0}^{\infty} b_n \cos n\pi r. \tag{23}$$

Velocity distribution is not thought to oscillate rapidly over the flow channel, due to constant developing force by almost uniform magnetic field. Therefore, in the Eq. (23), Fourier expansion can be approximated in terms of finite numbers of its polynomials under the assumption of no rapid oscillation of higher harmonics according to radial coordinate[13]. That is, Eq. (23) is represented by first three terms satisfying boundary conditions Eq. (22) as follows:

$$\begin{aligned}
\psi(r) &\simeq b_0(1 - \cos 2\pi\zeta) + b_1(\cos\pi\zeta - \cos 3\pi\zeta)\\
&\quad + a_1(\sin\pi\zeta - \frac{1}{3}\sin 3\pi\zeta)\\
&= b_0\phi_0 + b_1\phi_1 + a_1\phi_2,
\end{aligned} \tag{24}$$

where $\zeta = r - r_a$. After applying operator L_{lm} defined by $L_{lm} = (\phi_l, L\phi_m)$, to Eq. (24) where $(\phi_l, \phi_m) = \int_0^1 \phi_l(\zeta)\phi_m(\zeta)d\zeta$ and $L \equiv \frac{\partial^4}{\partial r^4} + \bar{A}\frac{\partial^3}{\partial r^3} + \bar{B}\frac{\partial^2}{\partial r^2} + \bar{C}\frac{\partial}{\partial r} + \bar{D}$, we have

$$b_o L\phi_o + b_1 L\phi_1 + a_1 L\phi_2 = 0 \tag{25}$$

Arranging Eq. (25) multiplied by ϕ_0, ϕ_1andϕ_2 yields linear simultaneous equations on the b_0, b_1 and a_1 as follows:

$$\begin{aligned} b_0 L_{00} + b_1 L_{01} + a_1 L_{02} &= 0 \\ b_0 L_{10} + b_1 L_{11} + a_1 L_{12} &= 0 \\ b_0 L_{20} + b_1 L_{21} + a_1 L_{22} &= 0 \end{aligned} \tag{26}$$

For the non trivial solution of Eq. (26), the determinant of a resultant coefficients matrix of the simultaneous equations should vanish.

$$\begin{vmatrix} L_{00} & L_{01} & L_{02} \\ L_{10} & L_{11} & L_{12} \\ L_{20} & L_{21} & L_{22} \end{vmatrix} = 0$$

Dispersion relation is found out from the determinant equation using the coefficients replaced by values at mean radius by narrow gap approximation. When the imaginary part of angular frequency sets to be zero in the dispersion relation obtained, a critical Reynolds number is found as function of Hartmann number along with U_0, R_0 and k as follows:

$$R_{e_{cr}} = \frac{4\pi^2 R_0^2 k^2 + 8\pi^4 + 2\pi H_a^2 - \frac{2\pi^2}{{r_{av}}^2}}{2\pi^2 U_0 R_0 k} \tag{27}$$

In the Fig. 2, critical Reynolds numbers are shown in log scales for a suitably designed pump with flowrate of 40 l/min. As seen in Fig. 2, critical Reynolds number $R_{e_{cr}}$ increases proportional to the square of Hartmann number H_a for the fixed wave number k. And it is shown that the flow becomes more stable due to a larger critical Reynolds number in case of perturbed wave with longer wave length($k \ll 1$). Fig. 3 shows the magnetic field effect on the stability for the long wave perturbation ($k \ll 1$), and then it is known that as a magnetic field increases, critical Reynolds number also increases and the stable flow is kept even in the higher speed. When Reynolds number of 13,666 on the pump with a flowrate of 40 l/min is compared with critical Reynolds number of order of more than 10^6 for the long wave perturbation, practical ALIP is predicted to operate stably by axisymmetric linear stability analysis from the very large difference of the two Reynolds numbers.

4 Conclusion

A dispersion relation has been obtained from the condition of a non-trivial solution for the ALIP by strong magnetic field like Eq.(27). And for putting Im(ω) zero, critical Reynolds numbers have been plotted on the Hartmann numbers perturbed wave numbers for a suitably designed ALIP with a flowrate of 40 l/min. As seen in

Fig. 2, critical Reynolds number($R_{e_{cr}}$) increases according to increasing Hartmann number(H_a) for a certain wave number(k). Thus, it is predicted that the effect of strong transverse magnetic field across annular channel gap suppresses the onset of instability.

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References

[1] R. S. Baker, *Mech. Eng.*, **15**, 1, 1964.

[2] M. Namba, *Toshiba Review*, **116**, 17, 1978.

[3] C. Yang and S. Kraus, *Nucl. Eng. Design*, **44**, 383, 1983.

[4] *Conduction Pumps Catalogues*, CA 21 - CA 22 - CA 81, Novatome, 1977.

[5] *Flat Induction Pumps Catalogues*, IP 121 - IP 201 - IP 202A - IP 501 - IP 653, Novatome, 1977.

[6] *Annular Induction Pumps Catalogues*, IA 124 - IA 251 - IA 501 - IA 1401, Novatome, 1977.

[7] H. J. Chung, S. Cho, S. Y. Chun, *J. of the Korean Nuclear Society*, **30**, NO. 4, 328, 1998.

[8] Herman Branover, *Magnetohydrodynamic Flow In Ducts*, John Wiley & Sons, New York, 1978.

[9] S. A. Nasar, Linear Motion Electric Machines, John Wiley & Sons, New York, 1976.

[10] Lawson P. Harris, *Hydromagnetic Channel Flows*, John Wiley & Sons, Inc., New York, 1960.

[11] P. Roverts, *An Introduction to Magnetohydrodynamics*, Longmans, London, 1967.

[12] S. V. Kozyrev and A.I. El'kin, *Magn. Gidrodin.*, **17**, NO. 4, 57, 1981.

[13] Zh. D. Genchev, *Magn. Gidrodin.*, **17**, NO. 4, 41, 1981.

[14] S. Yu. Reutskii, *Magn. Gidrodin.*, **23**, NO. 1, 121, 1987.

[15] Zh. N. Kaliev, *Magn. Gidrodin.*, **28**, NO. 2, 116, 1992.

[16] F. B. Hilderbrand, *Advanced Calculus for Applications*, Prentice-Hall, New Jersey, 1976.

[17] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, New York, 1961.

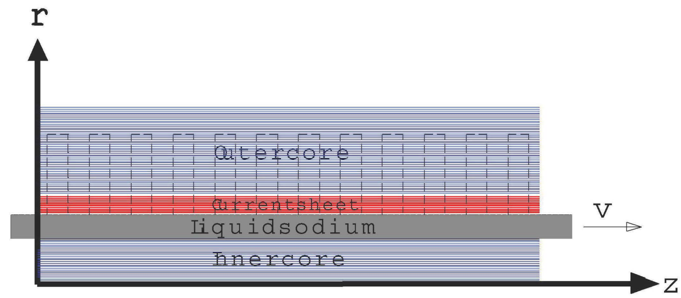

Fig. 1. Theoetical Model for ALIP

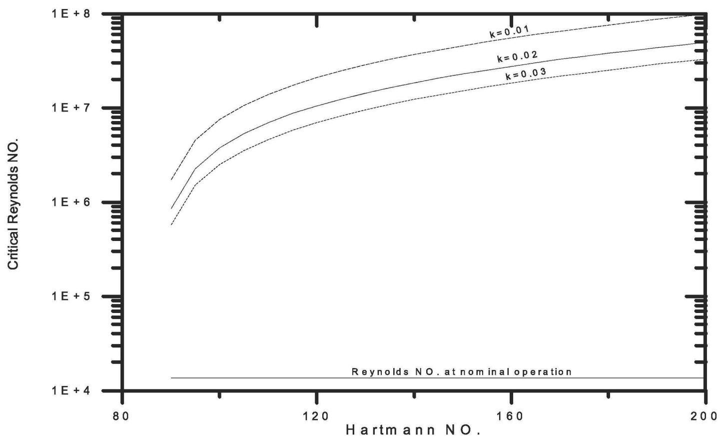

Fig. 2. Critical Reynolds number on the different wave numbers according to Hartmann number compared with Reynolds number obtained at the nominal operation of the pump with flowrate of 40 l/min

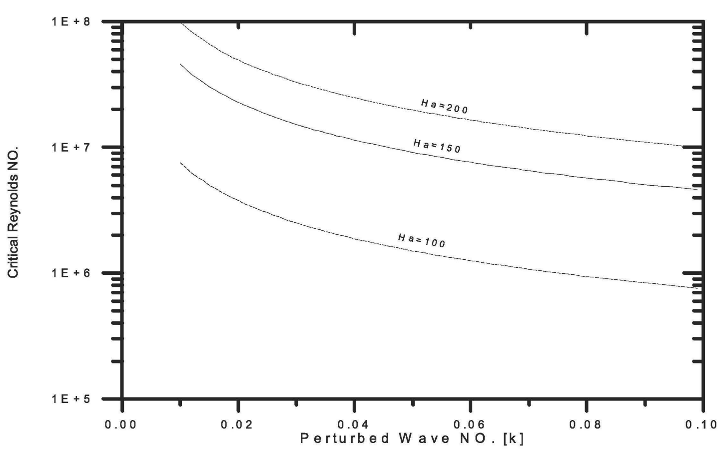

Fig. 3. Critical Reynolds number on the different Hartmann numbers according to perturbed wave number of long wave length